Research Statement

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1 Introduction

I am a combinatorialist with a wide interest in symmetric functions and hook length generalizations. My primary focus is Jack symmetric functions. Many problems I work on come from geometry and have a probabilistic flavor. I appreciate naive methods and take an experimental approach to most of my work. This involves collecting data in order to conceptualize ideas and describe potential conjectures, usually in SageMath.

My main work of late has been a problem rooted in geometry. In [5] and [6], Maulik, Nekrasov, Okounkov, and Pandharipande (hereafter, MNOP) share the *equivariant vertex measure*. My goal is to describe this measure on plane partitions in terms of combinatorial objects that are already well known. Indeed, I have already worked out the two-dimensional version of this problem: the *equivariant edge measure* is the Jack Plancherel measure on Young diagrams (up to sign) [7].

In another vein, I have proven several computations of Jack scalar products. I intend to expand and use these to potentially lead to a generalization of the *weighted branching rule for hook lengths* (WHL) of Ciocan-Fontanine, Konvalinka, and Pak [1]. Along the way, I have come across some interesting probability distributions on Young tableaux relating to Jack functions. I am interested in imbuing them with a hook-walk style proof as in the work of Greene, Nijenhuis, and Wilf [2, 3].

The next section of this statement goes over necessary background information and establishes conventions. I will then proceed to introduce my own work as well as potential directions for future work. Proofs are omitted in what follows, but I will provide this information upon request.

2 Background

For background information, I follow Richard Stanley's excellent exposition [8, 9].

2.1 Partitions

An integer partition λ is a non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ which contains only finitely many non-zero entries. Each entry λ_i in λ is called a *part* of λ. A partition λ may also be written as $\lambda = (1^{m_1}, 2^{m_2}, \cdots)$ where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i. The *size* of λ, denoted $|\lambda|$ is

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots .$$

The statement $|\lambda| = n$ can also be written as $\lambda \vdash n$ or λ *partitions* n. The *length* of λ, denoted $\text{length}(\lambda)$, is the largest i such that $\lambda_i \neq 0$.

2.1.1 Young Diagrams

It is often useful to have a visual representation for partitions. The *Young* (or *Ferrers*) *diagram* for a partition λ (in English notation) is the set of squares with matrix coordinates (i, j) where

$$1 \leq i \leq \text{length}(\lambda), \quad 1 \leq j \leq \lambda_i.$$

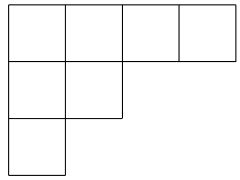

Figure 1: The Young diagram for the partition $\lambda = (4, 2, 1)$. The length of λ is 3 and $\lambda \vdash 7$.

See Figure 1. In the remainder of this document, I identify a partition λ with its Young diagram.

Partitions can be partially ordered via containment of Young diagrams. More formally, $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. This partially ordered set is called *Young's Lattice*. A *standard Young tableau* is a filling of each box of a shape λ with the integers in the set $\{1, 2, \ldots, |\lambda|\}$ such that each row increases rightward and each column increases downward. See Figure 2.

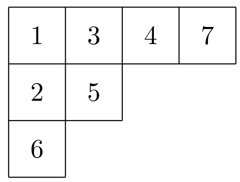

Figure 2: A standard Young tableau of shape $\lambda = (4, 2, 1)$.

2.1.2 Hook Lengths

Given a cell (i, j) in a shape λ, define the *arm length* of (i, j) as

$$a_\lambda((i, j)) = \lambda_i - j.$$

The *leg length* of a cell (i, j) is

$$\ell_\lambda((i, j)) = \lambda'_j - i$$

where λ'_j is the length of the j^{th} column of λ. In other words, the arm length (resp. leg length) of (i, j) is the number of boxes directly to the right (resp. below) (i, j). The *hook length* of a cell (i, j) in λ is

$$h_\lambda((i, j)) = a_\lambda((i, j)) + \ell_\lambda((i, j)) + 1.$$

We can omit the subscript λ when it is obvious from context.

2.2 Symmetric Functions

I follow the conventions of Richard Stanley in [8] and [9] for various bases of Λ, the vector space of symmetric functions. Specifically,

- the monomial symmetric function corresponding to λ is denoted by m_λ,
- the power sum symmetric function corresponding to λ is denoted by p_λ, and
- the Schur symmetric function corresponding to λ is denoted by s_λ.

2.3 Jack Symmetric Functions

Jack functions are homogeneous symmetric functions in $x_1, x_2, x_3, \ldots; t$ indexed by integer partitions. I will use J_λ for the (integral form) Jack polynomial corresponding to the partition λ. The variable t is called

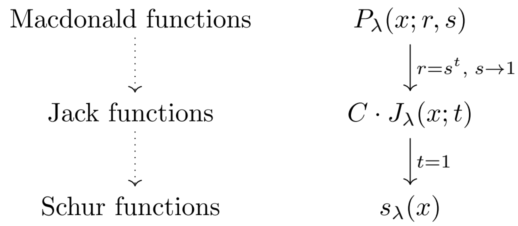

Figure 3: The relationship between Macdonald functions, Jack functions, and Schur functions. The scalar C is an explicit rational function in t that depends on λ.

the Jack parameter. The set of Jack symmetric functions form a basis for $\Lambda \otimes \mathbb{Q}(t)$. Moreover, Jack functions "sit between" Macdonald functions and Schur functions in terms of generality. See Figure 3.

Jack symmetric functions are orthogonal with respect to the scalar product $\langle -, - \rangle$ on $\Lambda \otimes \mathbb{Q}(t)$ given by

$$\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda,\mu} z_\lambda t^{\text{length}(\lambda)}$$

where p_λ is the power sum symmetric function corresponding to λ and

$$z_\lambda = 1^{m_1(\lambda)} \cdot 2^{m_2(\lambda)} \cdot 3^{m_3(\lambda)} \cdot \dots \cdot m_1(\lambda)! \cdot m_2(\lambda)! \cdot m_3(\lambda)! \cdot \dots \,.$$

Note that when $t = 1$, this is the standard scalar product on symmetric functions. The notation $j_\lambda := \langle J_\lambda, J_\lambda \rangle$ is used for the nontrivial pairing of a Jack functions with itself. There are two usual t-generalizations of the notion of hook length.

Definition 1 ([8], page 95)**.** *The upper and lower hook lengths of a cell $\Box \in \lambda$ are given by*

$$h^*_\lambda(\Box) = t(a(\Box) + 1) + \ell(\Box),$$
$$h^\lambda_*(\Box) = t(a(\Box)) + 1 + \ell(\Box)$$

respectively.

In a sense, this is just the standard hook length except that arm boxes are worth t instead of 1. The upper and lower hook lengths differ in what weight we assign the pivot box: the upper hook length assigns the pivot box t and the lower hook length assigns the pivot box 1.

Definition 2 ([4])**.** *The Jack Plancherel measure is a probability measure on partitions of n given by*

$$w_{\text{Jack}}(\lambda) = \prod_{\Box \in \lambda} \frac{1}{h^*_\lambda(\Box) h^\lambda_*(\Box)} \tag{1}$$

Much like for Schur symmetric functions, there is a Pieri rule for Jack symmetric functions ([8], page 98). For the purposes of this document, it suffices to present just a special case of the Jack Pieri rule.

Corollary 3 (Pieri Rule for Jack Symmetric Functions, One Box Version)**.** *Suppose $\lambda \vdash n$, $\mu \vdash (n-1)$, and $\mu \subseteq \lambda$. Then*

$$\langle J_\mu J_1, J_\lambda \rangle_t = t \cdot j_\mu \cdot \prod_{\Box \in \text{Row}(\lambda/\mu)} \frac{h^*_\lambda(\Box)}{h^*_\mu(\Box)} \prod_{\Box \in \text{Col}(\lambda/\mu)} \frac{h^\lambda_*(\Box)}{h^\mu_*(\Box)} \tag{2}$$

where $\text{Col}(\lambda/\mu)$ (resp. $\text{Row}(\lambda/\mu)$) is the set of boxes that appear in the same column (resp. row) of λ as the corner box λ/μ excluding the box λ/μ itself.

3 Jack Combinatorics of the Equivariant Vertex Measure

In a forthcoming preprint, Ben Young and I demonstrate that two seemingly unrelated measures on Young diagrams are the same.

As in [5] and [6], let Q and $\overline{Q}$ be generating functions for the cells (i,j) in a partition λ in the variables r and s:

$$Q := \sum_{(i,j)\in\lambda} r^i s^j, \qquad \overline{Q} := \sum_{(i,j)\in\lambda} r^{-i} s^{-j}.$$

Moreover, define a Laurent polynomial F in r, s:

$$F(\lambda) = -Q(\lambda) - \frac{\overline{Q}(\lambda)}{rs} + \frac{Q(\lambda)\overline{Q}(\lambda)(1-r)(1-s)}{rs}.$$

Given an index set A and a Laurent polynomial $G = \sum_{(i,j)\in A} c_{i,j} r^i s^j$ in the variables r and s with no constant term, define

$$\mathrm{swap}(G) := \mathrm{swap}\left(\sum_{(i,j)\in A} c_{i,j} r^i s^j\right) = \prod_{(i,j)\in A} (iu - jv)^{c_{i,j}}$$

Definition 4. *The equivariant edge measure of a partition λ is*

$$w_{\mathrm{MNOP}}(\lambda) := \mathrm{swap}(F(\lambda)). \tag{3}$$

Our result is that (up to sign) this measure is the Jack Plancherel measure.

Theorem 5 (P. - Young, 2024, [7])**.** *The Jack Plancherel measure of a partition λ is the same as the equivariant edge measure of λ up to a sign, i.e.*

$$w_{\mathrm{Jack}}(\lambda) = -w_{\mathrm{MNOP}}(\lambda). \tag{4}$$

I'm interested in continuing this line of inquiry by exploring the three dimensional case of this problem. Let π be a plane partition. Let F' and swap$'$ be three dimensional generalizations of F and swap. I hope to give the *equivariant vertex measure* $\mathrm{swap}'(F'(\pi))$ some combinatorial explanation and ascertain whether it is some combinatorial object that is already well known.

4 The Work of Ciocan-Fontanine, Konvalinka, and Pak

There is a curious connection between Jack functions and the *weighted branching rule for hook lengths* (WHL) of Ciocan-Fontanine, Konvalinka, and Pak.

Theorem 6 ([1], page 2, The Weighted Branching Rule for Hook Lengths)**.**

$$\sum_{\textit{corners } (r,s)\in\lambda} x_r y_s \prod_{i=1}^{r-1}\left(1 + \frac{x_i}{x_{i+1}+\cdots+x_r+y_{s+1}+\cdots+y_{\lambda_i}}\right) \cdot \prod_{j=1}^{s-1}\left(1+\frac{y_j}{x_{r+1}+\cdots+x_{\lambda'_j}+y_{j+1}+\cdots+y_s}\right) = \sum_{(i,j)\in\lambda} x_i y_j \tag{WHL}$$

The authors of [1] make no mention of Jack functions in their description and exploration of this identity. However, I have found such a connection.

Proposition 7 (P., 2023)**.** *Specializing $x_k = t$ and $y_k = 1$ in (WHL) for all k implies the one-box version of the Jack Pieri rule.*

I am interested in generalizing (WHL). In particular, I'd like to find a generalization of the one box version of the Jack Pieri rule that yields a pullback of (WHL).

$$\begin{array}{ccc} \text{Pullback} & \longrightarrow & \text{One Box Pieri Generalization} \\ \downarrow & & \downarrow \\ \text{(WHL)} & \xrightarrow{\text{sp: } x_i = 1,\ y_j = t} & \text{One Box Pieri Rule} \end{array}$$

There are several options to consider as the top right entry in this diagram since there are several ways to generalize the idea of "adding one box to a diagram." For example, in terms of Schur symmetric functions, both the Pieri rule and the Murnaghan-Nakayama rule specialize to the one-box Pieri rule. There may be other possibilities worth considering in the Jack case as well.

5 Probability Distributions and Hook Walks

Lemma 8 (P.-Young, 2023)**.** *Suppose the size of λ is n. Then*

$$\sum_{\mu \vdash n-1} j_\mu^{-1} \langle J_\mu J_1, J_\lambda \rangle_t = nt. \tag{5}$$

From Lemma 8, I have derived a probability distribution on Young tableaux.

Theorem 9 (P., 2024)**.** *Let $\lambda \vdash n$ and let $\varnothing = \lambda^0 \subseteq \lambda^1 \subseteq \lambda^2 \subseteq \cdots \subseteq \lambda^{n-1} \subseteq \lambda$ be a saturated chain in Young's lattice beginning at the empty partition $\varnothing$ and ending at λ. (In other words, $\lambda^i \lessdot \lambda^{i+1}$.) Then*

$$1 = \sum_{\varnothing = \lambda^0 \subseteq \lambda^1 \subseteq \lambda^2 \subseteq \cdots \subseteq \lambda^{n-1} \subseteq \lambda} \frac{1}{n! t^n} \prod_{i=0}^{n-1} \frac{\langle J_{\lambda^i} J_1, J_{\lambda^{i+1}} \rangle_t}{j_{\lambda^i}}. \tag{6}$$

When t is positive, the summand corresponding to $\varnothing = \lambda^0 \subseteq \lambda^1 \subseteq \lambda^2 \subseteq \cdots \subseteq \lambda^{n-1} \subseteq \lambda$ on the right hand side in (6) can be viewed as the probability in this distribution of selecting the standard Young tableau generated by putting i into the box λ^i/λ^{i-1}.

I plan to extract a combinatorial interpretation from this probability distribution. My next steps in this process will be exploring a small example by hand and then running a simulation in SageMath. Ideally, I will develop a hook walk that yields this distribution of tableaux in the style of [2] and [3].

Ben Young and I also discovered a related lemma that yields a similar probability distribution.

Lemma 10 (P.-Young, 2024)**.** *Fix a partition $\mu \vdash n-1$. Then*

$$\sum_{\lambda : \mu \nearrow \lambda} \frac{\langle J_\mu J_1, J_\lambda \rangle}{j_\lambda} = 1. \tag{7}$$

Our initial experiments suggest that the resulting distribution can be achieved combinatorially by the most straightforward Jack generalization of the outside hook walk presented in [3]: Start at a cell (i, j) outside of λ such that $i \geq \lambda'_1$ and $j \geq \lambda_1$. Then move to a given arm cell with probability $\frac{t}{ta(\square)+\ell(\square)}$ and move to a given leg cell with probability $\frac{1}{ta(\square)+\ell(\square)}$. See Figure 4. Repeat this until an outside corner of the diagram is reached. Add this cell to the diagram and then repeat this entire process with the enlarged diagram.

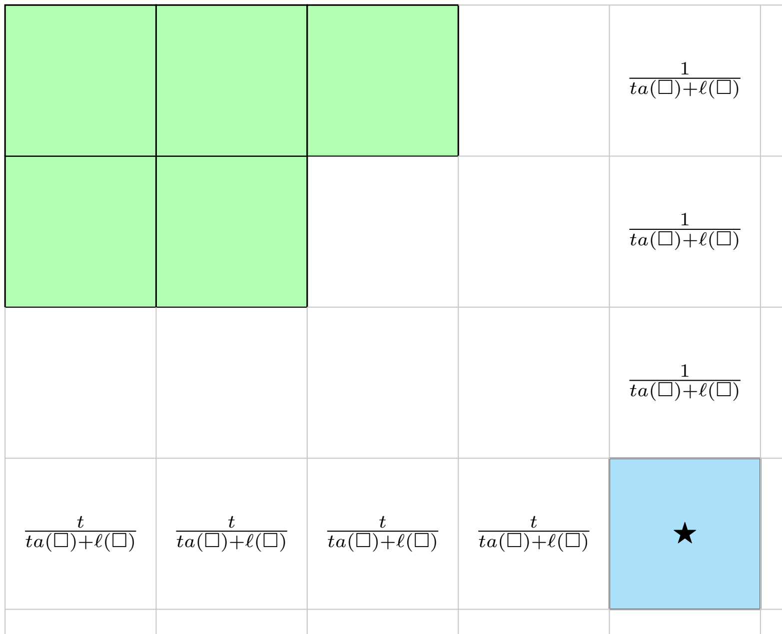

Figure 4: Starting from the cell indicated with a star outside of the Young diagram, the probability of moving to a given cell in the starred cell's arm or leg is given inside that cell.

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