Partitions and Hook Walk Algorithms

Kyla Pohl

ADSE Lane Community College STEM Seminar

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About Me

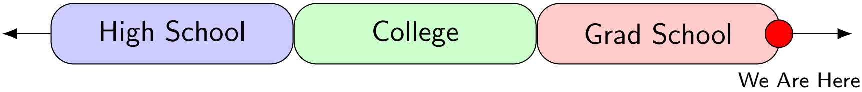
- I am a 6th year mathematics PhD student at the University of Oregon.
- The math that I do is called "combinatorics."
- I'm currently applying for jobs as a postdoc or a math professor.

My Timeline

High School:
Battle Ground High School
Hometown: Battle Ground, WA
Years: 2011 - 2015

College:
St. Olaf College
Northfield, MN
Major/Minor: Math / Japan Studies
Years: 2015 - 2019

Grad School:
University of Oregon
Degree: PhD, Mathematics
Years: 2019 -



Partitions

Definition

An *partition* of a positive integer n is a way to write n as a sum of other positive integers listed from biggest to smallest.

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Example

One partition of 7 is $4 + 2 + 1$.

Partitions

Example

Here is a list of all the partitions of 4.

$$4$$
$$3+1$$
$$2+2$$
$$2+1+1$$
$$1+1+1+1$$

Partitions

We label partitions with Greek letters. We can also avoid writing the plus sign and just list all the parts inside parentheses. So for example, we could write $4 = 3 + 1$ as $\lambda = (3, 1)$.

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Definition

The *size* of λ is the sum of all of its parts.

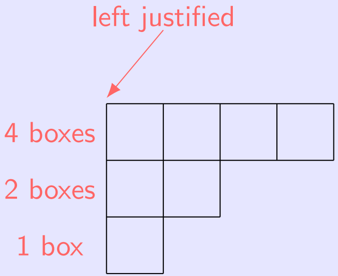
The size of $\lambda = (3, 1)$ is 4 since $3 + 1 = 4$.

Young Diagrams

Let's draw pictures of partitions. For each part of a partitions λ, draw that many boxes in a row, left justified. We'll add a new row to our picture for each part.

Example

Let $\lambda = (4, 2, 1)$. When we draw a picture of λ as described above, we get the following.



Young Diagrams

Example

Let's draw the Young diagrams for all partitions of 4.

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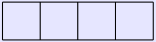
$$\lambda = (4)$$



Young Diagrams

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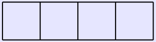
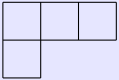
$\lambda = (4)$ $\mu = (3, 1)$





Young Diagrams

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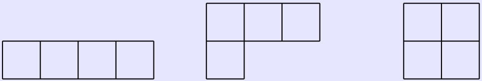
$\lambda = (4) \qquad \mu = (3,1) \qquad \nu = (2,2)$



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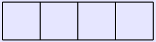
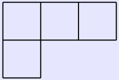
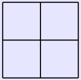
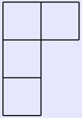






$\sigma = (2, 1, 1)$



Young Diagrams

Example

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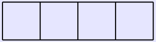

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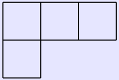

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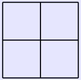

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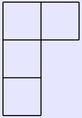
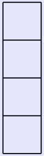
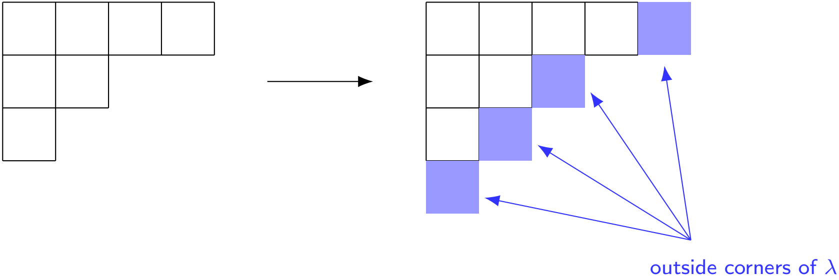


$\beta = (1, 1, 1, 1)$



New Diagrams from Old

If I give you a Young diagram λ and one extra square, you can build a new diagram by placing the new square in one of the *outside corners* of λ.



New Diagrams from Old

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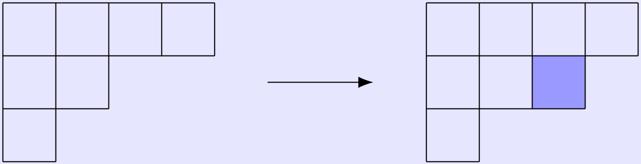

$\lambda = (4, 2, 1)$ $\mu = (4, 3, 1)$

Something from Nothing

So what would happen if we started with the partition (1)

Something from Nothing

So what would happen if we started with the partition (1) and then added a square

Something from Nothing

So what would happen if we started with the partition (1) and then added a square and added another square

Something from Nothing

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So what would happen if we started with the partition (1) and then added a square and added another square and then another and so on?

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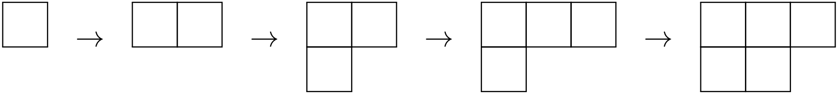
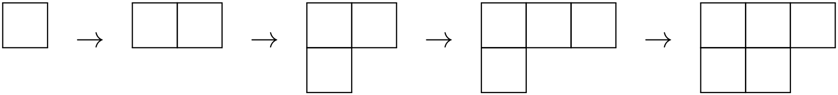


Something from Nothing

So what would happen if we started with the partition (1) and then added a square and added another square and then another and so on?



We'll end up with a new partition that's as big as we like.

Something from Nothing

So far, we've allowed ourselves to put our new box in any outside corner that we choose.

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Let's go a step further and put some rules on how we add boxes.

Something from Nothing

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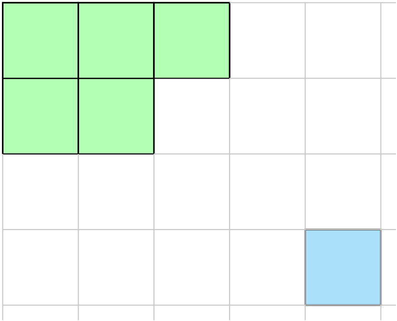
Let's go a step further and put some rules on how we add boxes.

We'll call the resulting process Random Rook.

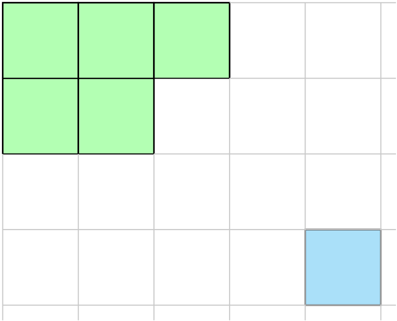
Rules of the Game



Rules:

- Start at a spot below and to the right of our diagram.

Rules of the Game



Rules:

- Start at a spot below and to the right of our diagram.
- Next, walk towards the diagram only taking steps left or up.

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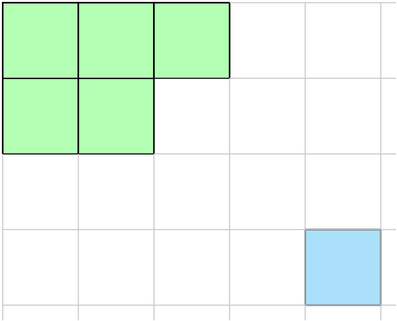
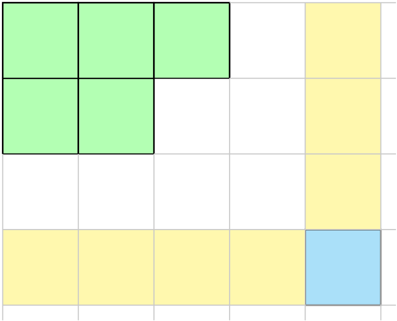


Rules:

- Start at a spot below and to the right of our diagram.
- Next, walk towards the diagram only taking steps left or up.
- When we step, we have an equal chance of landing on any square that is directly to the left of or above the current square.

Rules of the Game



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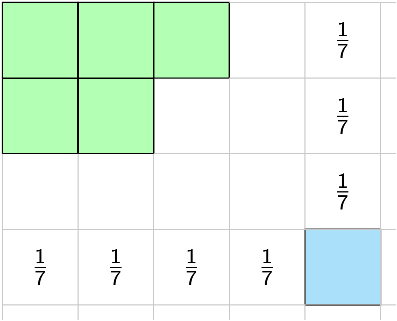

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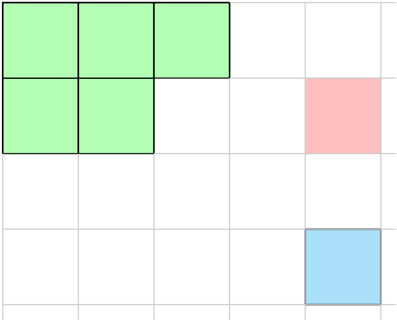

- Say we land on the pink square.
- Now repeat the steps we just did again.
- Keep repeating them until we've reached an outside corner.

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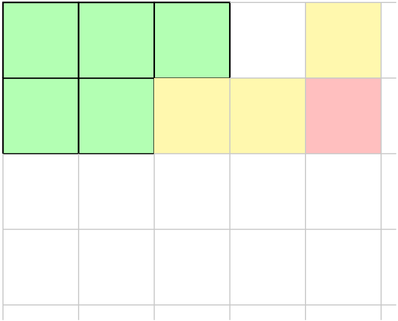

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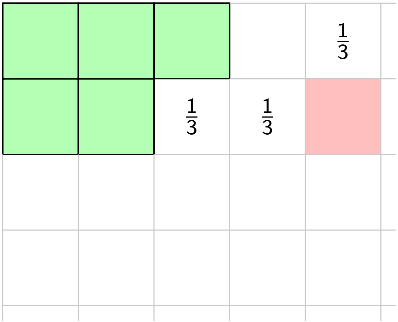

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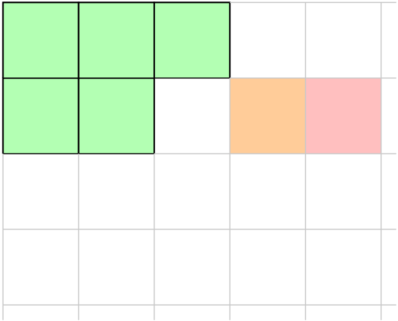

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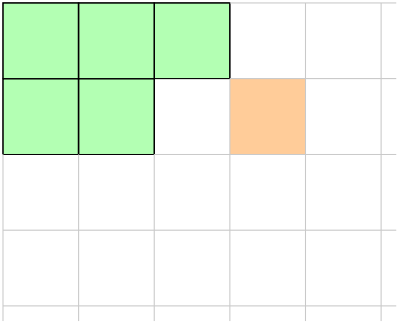

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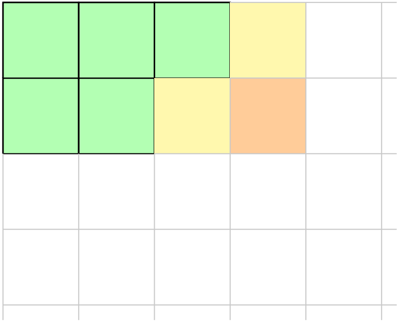

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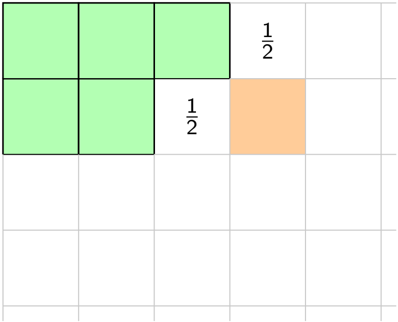

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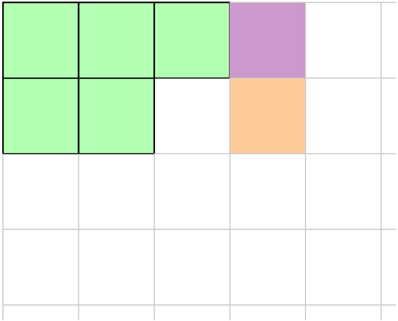

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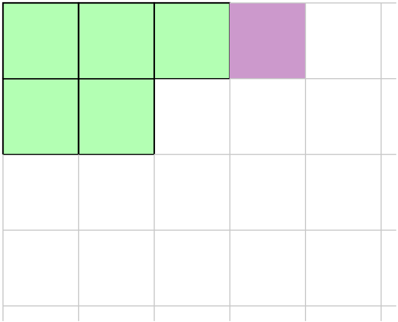

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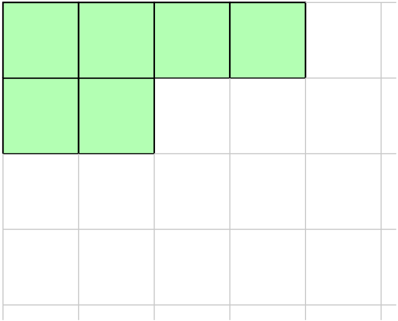
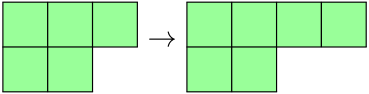


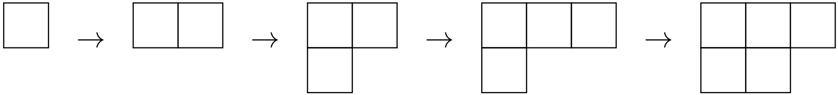
Rules of the Game

Congratulations! We've just turned the partition $(3, 2)$ into the partition $(4, 2)$ by playing Random Rook!



Once More

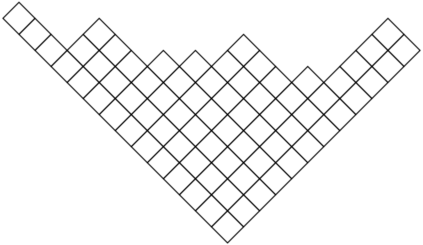
Now let's start with the partition (1) and add squares one at a time by playing Random Rook repeatedly. This is exactly what we did earlier



except now we're not choosing where to add the extra box ourselves – instead, we're following the Rules of Random Rook.

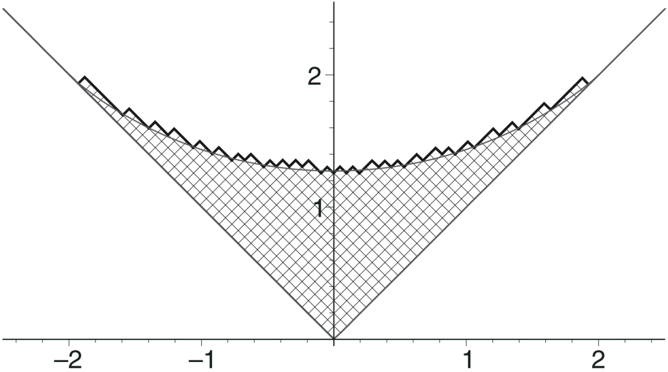
The Limit Shape

If we do this for a long time, making a bigger and bigger shape and then turn it so it's resting on its point, like so,



then the ragged edge of the diagram will always make the same approximate shape as the diagram increases in size!

The Limit Shape



What do I think about?

I ask questions about what happens when we change the rules of Random Rook slightly.

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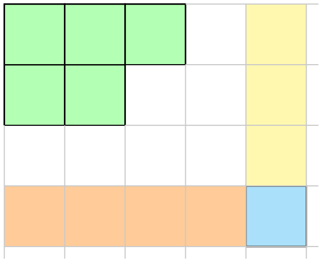
For example, what happens when we prefer to go to the left instead of up at each step?



What kinds of shapes do we usually end up building?

Thank you!