Jack Combinatorics of the Equivariant Edge Measure

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Introductory Notions

Definition

A *plane partition* is an array $\pi = (\pi_{i,j})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e. finitely many nonzero entries) and is weakly decreasing in the rows and columns.

Example

3	2	2
1	1	

Introductory Notions

Definition

The sum of all of the entries in a plane partition π is the *size* of π. We denote this $|\pi|$.

Theorem (MacMahon)

The number of plane partitions with size n is the coefficient of q^n in

$$M(q) = \prod_{i \geq 1} \left(\frac{1}{1 - q^i} \right)^i.$$

Motivation

Definition

Define

$$Q(\pi) = \sum_{(i,j,k)\in\pi} r^i s^j t^k$$

$$\overline{Q}(\pi) = \sum_{(i,j,k)\in\pi} r^{-i} s^{-j} t^{-k}.$$

Example

3	2	2
1	1	

Given the plane partition π as before,

$$Q = 1 + r + r^2 + s + rs + t + rt + r^2 t + t^2$$

$$\overline{Q} = 1 + r^{-1} + r^{-2} + s^{-1} + r^{-1}s^{-1} + t^{-1} + r^{-1}t^{-1} + r^{-2}t^{-1} + t^{-2}.$$

Motivation

Definition

From Q and $\overline{Q}$ define

$$F = Q - \frac{\overline{Q}}{rst} + Q\overline{Q}\frac{(1-r)(1-s)(1-t)}{rst} = \sum_{i,j,k} c_{ijk} r^i s^j t^k.$$

Definition

The *equivariant vertex measure* is obtained by "swapping the roles of addition and multiplication" in F:

$$\mathbf{w}(\pi) = \prod_{i,j,k} (iu + jv + kx)^{-c_{ijk}}.$$

We use the variables u, v, and x instead of r, s, and t post-swap.

Motivation

Maulik, Nekrasov, Okounkov and Parharipande give a generating function for $\mathbf{w}(\pi)$ in their 2005 paper.

Theorem (MNOP, 2005)

$$Z := \sum_{\pi} \mathbf{w}(\pi) q^{|\pi|} = M(q)^{-\frac{(u+v)(v+x)(x+u)}{uvx}}$$

Example (in lieu of proof...)

Consider the unique plane partition π of size 1.
Only the $i = 1$ term of $M(q)$ yields any q^1 terms:

$$[q^1](1-q)^{-\frac{(u+v)(v+x)(x+u)}{uvx}} = \frac{(u+v)(v+x)(x+u)}{uvx}.$$

$$\mathbf{w}(\pi) = (-v-x)(-u-x)(-u-v)(-x)^{-1}(-v)^{-1}(-u)^{-1} = \frac{(v+x)(u+x)(u+v)}{uvx}.$$

Motivation

The proof of Z is geometric. One could hope for a combinatorial proof; however, that is currently out of reach.

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The subject of this talk is a warm-up problem for this: the same problem one dimension down.

In two dimensions...

Definitions

$$Q_2(\lambda) = \sum_{(i,j)\in\lambda} r^i s^j$$

$$\overline{Q}_2(\lambda) = \sum_{(i,j)\in\lambda} r^{-i} s^{-j}$$

$$F_2(\lambda) = F_2 = -Q_2 - \frac{\overline{Q}_2}{rs} + Q_2 \overline{Q}_2 \frac{(1-r)(1-s)}{rs} = \sum_{i,j} c_{ij} r^i s^j$$

In two dimensions. . .

Example

Note that Q_2 assigns a monomial to each box in a shape λ which describes the (matrix) coordinates of the box.

1	r	r^2
s	rs	

Next, we define an operation on Laurent polynomials which switches the roles of addition and multiplication.

Definition

Let $G = \sum_{i,j} d_{i,j} r^i s^j$ be a Laurent polynomial in the variables r and s with no constant term. Then define the *swap* of G to be

$$\text{swap}(G) = \prod_{i,j} (iu - jv)^{d_{i,j}}.$$

Things to note: sign convention, variable changes

Definition

The *equivariant edge measure* is

$$w_{\text{MNOP}}(\lambda) := \text{swap}(F_2(\lambda)) = \prod_{i,j} (iu - jv)^{c_{ij}}.$$

So what is w_{MNOP}?

Reminder: Our goal is to give some combinatorial meaning to w_{MNOP}.

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It turns out that w_{MNOP} is (up to convention) the Jack Plancherel measure.

Jack Plancherel Measure

Theorem (Jack Plancherel Measure)

Set

$$h^*(i,j) = u + u(\lambda_i - j) + v(\lambda'_j - i)$$
$$h_*(i,j) = v + u(\lambda_i - j) + v(\lambda'_j - i).$$

We have

$$1 = \sum_{\lambda \vdash n} \frac{n!(uv)^n}{\prod_{\square \in \lambda} h^*(\square) h_*(\square)}.$$

Jack Plancherel Measure

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$$1 = \sum_{\lambda \vdash n} \frac{n!(uv)^n}{\prod_{\square \in \lambda} h^*(\square) h_*(\square)}.$$

For our result, we need a slightly different version of this.

Definition

Define
$$w_{\text{Jack}}(\lambda) = \frac{1}{\prod_{\square \in \lambda} h^*(\square) h_*(\square)}.$$

Main Result

Theorem (P.-Young)

We have

$$w_{Jack} = -w_{MNOP}.$$

The notion of w_{MNOP} comes from areas of algebraic geometry (specifically, Hilbert schemes and Donaldson-Thomas theory) in which Jack polynomials frequently arise. However, this particular connection appears to be new.

Back to Motivations

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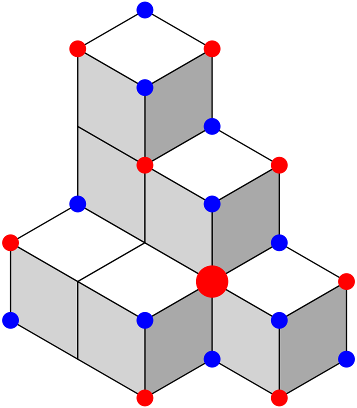
It's much more difficult, but we now know that $\mathbf{w}(\pi)$ is an analogue for **plane partitions** in the Jack Plancherel measure.



Thank you!