The Hilbert Scheme of Points in the Plane

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1 Introduction

These notes are based on Haiman's 1998 paper "t,q-Catalan Numbers and the Hilbert Scheme." We follow it closely starting at the beginning of Section 2 all the way through Proposition 2.4. Red text indicates uncertainty on my part. Feel free to contact me kyla.pohl@gmail.com with clarification if you are inclined to do so.

Let $\Bbbk$ be a field of characteristic 0. We use the French convention for Young diagrams and zero-index the cells in this note.

2 Definitions

Definition 1. *The* punctual Hilbert scheme of the plane, $\mathbb{H}^n = Hilb^n(\mathbb{A})$, *is the set of all ideals I such that*

$$\dim_{\Bbbk}(\Bbbk[x,y]/I) = n. \tag{1}$$

Definition 2. *Given $\mu \vdash n$, let*

$$\mathcal{B}_\mu = \{x^h y^k \mid (h,k) \in \mu\}. \tag{2}$$

Example 3. *Let $\mu = (4,4,2,2)$. Then $\mathcal{B}_\mu$ contains all of the following monomials.*

$$\begin{array}{cccc} x^3 & x^3y & & \\ x^2 & x^2y & & \\ x & xy & xy^2 & xy^3 \\ 1 & y & y^2 & y^3 \end{array}$$

Example 4. *Let $\mu = (2,1)$. Then $\mathcal{B}_\mu$ contains all of the following monomials.*

$$\begin{array}{cc} x & \\ 1 & y \end{array}$$

Definition 5. *Define*

$$U_\mu = \{I \in H^n \mid \mathcal{B}_\mu \text{ spans } \Bbbk[x,y]/I\}. \tag{3}$$

Example 6. *We use our previous example involving $\mu = (2,1)$. The ideals*

- $\langle x^2, xy, y^2 \rangle$,

- $\langle x^2 - y, xy, y^2 \rangle$,
- $\langle x^2, xy, y^2 - x^2 \rangle$

are in $U_{(2,1)}$.

Since $\dim_{\Bbbk}(\Bbbk[x,y]/I) = n$ for $I \in H^n$ by definition, $\mathcal{B}_\mu$ is a basis of $\Bbbk[x,y]/I$. Hence for each monomial $x^r y^s$ and ideal $I \in U_\mu$ there is a unique expansion

$$x^r y^s \equiv \sum_{(h,k)\in\mu} c^{rs}_{h,k}(I) x^h y^k \pmod{I}, \tag{4}$$

whose coefficients depend on I and thus define a collection of functions $c^{rs}_{hk} : U_\mu \to \Bbbk$.

Proposition 7. *The sets* U_μ *are open affine subvarieties which cover* H^n. *The affine coordinate ring* $\mathcal{O}_{U_\mu}$ *is generated by the functions* c^{rs}_{hk} *for* $(h,k) \in \mu$ *and all* (r,s).

Proof. The sets U_μ cover H^n because: for every ideal I in a polynomial ring there is a basis $\mathcal{B}$ modulo I consisting of monomials such that every divisor of a monomial in $\mathcal{B}$ is also in $\mathcal{B}$. (This is a fact of Gordan, often attributed to Gröbner basis theory.) For $I \in H^n$ it is clear that such a basis must be $\mathcal{B}_\mu$ for some $\mu \vdash n$.

We now blackbox some facts. Let M_N be the set of all monomials in x and y of degree at most N. We write $G^n(\Bbbk M_N)$ for the Grassmann variety of n-dimensional quotients of the linear span of M_N. There is an injective map

$$H^n \to G^n(\Bbbk M_N) \tag{5}$$

which defines the structure of H^n as a scheme. The sets U_μ are preimages under this embedding of standard affines on $G^n(\Bbbk M_N)$ and the standard coordinates on these affines reduce to the functions c^{rs}_{hk}. The image of U_μ is closed in the corresponding standard affine on $G^n(\Bbbk M_N)$, so the functions c^{rs}_{hk} generate $\mathcal{O}_{U_\mu}$. ☐

For each $I \in H^n$, the scheme $S = \mathit{Spec}(\Bbbk[x,y]/I)$ has a finite number of points. If we assign each point p in S a multiplicity m_p equal to the length of the local ring $\mathcal{O}_{p,S} = (\Bbbk[x,y]/I)_p$ then these multiplicities sum to n. In this way, we associate with I an n-element multiset $\pi(I) \in \mathbb{A}^2$.

Example 8. *Let* $I = (x^3, y) \subseteq \Bbbk[x,y]$. *On what points does this ideal vanish? Only at* $(0,0)$. *So we say the subscheme at* I *is concentrated at* $(0,0)$. *Note that* $\Bbbk[x,y]/I$ *has basis* $\{1, x, y\}$ *so it is dimension 3 as a* $\Bbbk$ *vector space. The length at the point* $(0,0)$ *is 3, so* $\pi(I)$ *is* $\{(0,0),(0,0),(0,0)\}$.

Example 9. *Let* $I = (x(x-1), y) \subseteq \Bbbk[x,y]$. *The points which vanish on* I *are* $(0,0)$ *and* $(1,0)$. *A basis for* $\Bbbk[x,y]/I$ *is* $\{1, x\}$. *The multiset* $\pi(I)$ *is* $\{(0,0),(1,0)\}$.

This next fact is blackboxed once again: The n-element multisets contained in $\mathbb{A}^2$ form an affine variety $\mathrm{Sym}^n(\mathbb{A}^2)$. The map $\pi : H^n \to \mathrm{Sym}^n(\mathbb{A}^2)$ is called the Chow morphism.

Proposition 10. *The Chow morphism* $\pi : H^n \to \mathit{Sym}^n(\mathbb{A}^2)$ *is a projective morphism. (In other words, it factors through projective space.)*

Proof. Omitted. ☐

3 The Torus Action

The two dimensional torus group

$$T^2 = \{(t,q) \mid t, q \in \Bbbk^*\} \tag{6}$$

acts algebraically on $\mathbb{A}^2$ by $(t,q)\cdot(\xi,\zeta) = (t\xi, q\zeta)$, or equivalently on $\Bbbk[x,y]$ by $(t,q)\cdot x = tx$ and $(t,q)\cdot y = qy$. There is an induced action of H^n which is given by $(t,q)\cdot c^{rs}_{hk} = t^{r-h}q^{s-k}c^{rs}_{hk}$. Why? Because (4) must remain invariant. Note $(t,q)\cdot I = \{p(t^{-1}x, q^{-1}y) \mid p(x,y) \in I\}$.

An ideal $I \in H^n$ is a T^2 fixed point if and only if I is spanned by monomials. Why? Such an ideal must be of the form

$$I_\mu = (x^h y^k \mid (h,k) \notin \mu) \tag{7}$$

for some partition μ of n. Note the the subscheme of $\mathbb{A}^2$ defined by such an ideal I_μ is concentrated at the origin, the sole T^2 fixed point of $\mathbb{A}^2$.

Lemma 11. *Every ideal $I \in H^n$ has a torus fixed point in the closure of its orbit. We have $\lim_{a\to 0} \lim_{b\to 0}(a,b)\cdot I = I_\mu$ for some partition μ of n.*

Proof. Omitted. $\square$

4 Main Theorem

Theorem 12. *The punctual Hilbert scheme H^n of $\mathbb{A}^2$ is smooth of dimension $2n$.*

Note: "Smooth" means the same dimension everywhere.

Note: This is special to the two dimensional setting. In three dimensions, we can't say much besides the parity.

Proof. It suffices to verify smoothness near each T^2 fixed point I_μ. Why? I don't know. This is probably from Lemma 11. Under $\pi : H^n \to \mathrm{Sym}^n(\mathbb{A}^2)$ we can see that the image $\pi(U_\mu)$ is dense in $\mathrm{Sym}^n(\mathbb{A}^2)$. So U_μ has dimension at least $2n$.

The maximal ideal I_μ in $\mathcal{O}_{U_\mu}$ is given by

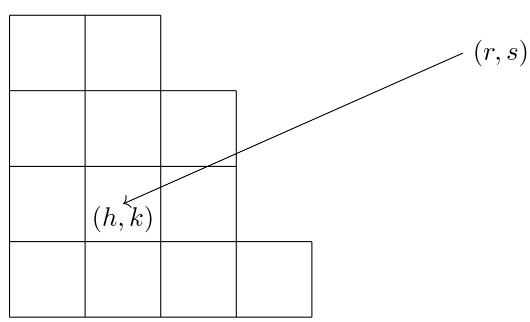
$$m = (c^{rs}_{hk} \mid (h,k) \in \mu, (r,s) \notin \mu). \tag{8}$$

Why? Well, for $(r,s) \in \mu$ we have $c^{rs}_{hk} = 0$ when $(h,k) \neq (r,s)$ and $c^{rs}_{rs} = 1$, so we omit these c^{rs}_{hk} from the ideal.

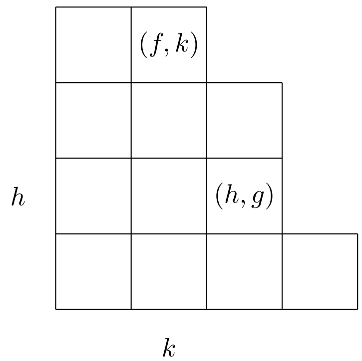
Plan: Find $2n$ of the coordinate functions c^{rs}_{hk} which span the cotangent space m/m^2. This will show that $\dim(m/m^2) = 2n$, so H^n is smooth at I_μ.

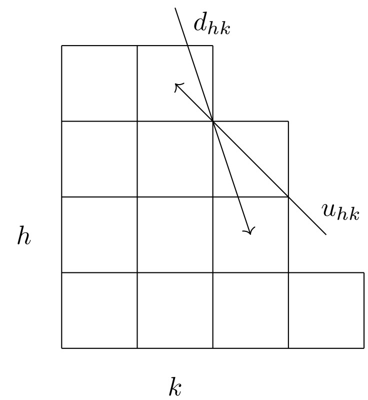
With this in mind, it is convenient to think of each c^{rs}_{hk} as an arrow from (r,s) to (h,k) overlaying the diagram μ.

Example 13.



We now single out two special coordinate functions at each square (h,k) in μ. Let (f,k) be the top square in column k and let (h,g) be the last square in row h.





Let $u_{hk} = c_{f,k}^{h,g+1}$ and $d_{hk} = c_{h,g}^{f+1,k}$. These will be our $2n$ spanning parameters for m/m^2. Multiply (4) through by x and then expand both sides by (4) again.

$$\sum_{(h,k)\in\mu} c_{h,k}^{r+1,s}(I)x^h y^k \equiv x^{r+1}y^s \tag{9}$$

$$\equiv x \cdot \sum_{(f,g)\in\mu} c_{f,g}^{r,s}(I)x^f y^g \tag{10}$$

$$\equiv \sum_{(f,g)\in\mu} c_{f,g}^{r,s}(I)x^{f+1} y^g \tag{11}$$

$$\equiv \sum_{(f,g)\in\mu} c_{f,g}^{r,s}(I)\left(\sum_{(h,k)\in\mu} c_{h,k}^{f+1,g}(I)x^h y^k\right) \pmod{I} \tag{12}$$

This yields

$$c_{h,k}^{r+1,s} = \sum_{(f,g)\in\mu} c_{f,g}^{r,s} c_{h,k}^{f+1,g}. \tag{13}$$

Modulo m^2, the terms on the RHS of (13) reduce to zero besides one: $c_{h-1,k}^{r,s}$. (And $c_{h-1,g}^{r,s} = 0$ if $h = 0$). So in m/m^2, we have

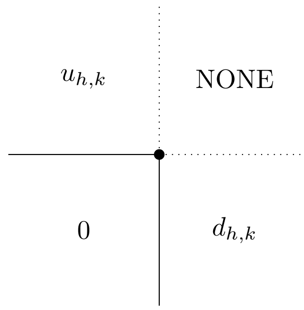
$$c_{h,k}^{r+1,s} = c_{h-1,k}^{r,s} \text{ (or 0 if } h = 0). \tag{14}$$

We can do something analogous if we multiply (4) by y. We get

$$c_{h,k}^{r,s+1} = c_{h,k-1}^{r,s} \text{ (or 0 if } k = 0). \tag{15}$$

Equations (14) and (15) say that we can slide arrows in our picture up, down, left, or right without their values modulo m^2 provided we keep the arrow head inside μ and the tail outside. More generally, as long as we kep the tail in the first quadrant and outside of μ, we may even move the head across the x- or y- axis. When this is possible, the value of the arrow is zero.

Which arrow survive? Only the $u_{h,k}$ and $d_{h,k}$. Northwest and west arrows are the $u_{h,k}$. Southwest and south arrows are the $d_{h,k}$. There are no north, east, or northeast arrows. All southwest arrows are zero.



Since we have found our $2n$ spanning elements, we can conclude that H^n is smooth as claimed. The previous lemma shows that H^n is connected. Smooth and connected implies irreducible.

$\square$