Jack Combinatorics of the Equivariant Edge Measure

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Introductory Notions

Definition

A *plane partition* is an array $\pi = (\pi_{i,j})_{i,j \geq 1}$ of nonnegative integers such that π has finite support (i.e. finitely many nonzero entries) and is weakly decreasing in the rows and columns.

Example

3	2	2
1	1	

Introductory Notions

Definition

The sum of all of the entries in a plane partition π is the *size* of π. We denote this $|\pi|$.

Theorem (MacMahon)

The number of plane partitions with size n is the coefficient of q^n in

$$M(q) = \prod_{i \geq 1} \left(\frac{1}{1 - q^i} \right)^i .$$

Motivation

Definition

Define

$$Q(\pi) = \sum_{(i,j,k)\in\pi} r^i s^j t^k$$

$$\overline{Q}(\pi) = \sum_{(i,j,k)\in\pi} r^{-i} s^{-j} t^{-k}.$$

Example

3	2	2
1	1	

Given the plane partition π as before,

$$Q = 1 + r + r^2 + s + rs + t + rt + r^2 t + t^2$$

$$\overline{Q} = 1 + r^{-1} + r^{-2} + s^{-1} + r^{-1}s^{-1} + t^{-1} + r^{-1}t^{-1} + r^{-2}t^{-1} + t^{-2}.$$

Motivation

Definition

From Q and $\overline{Q}$ define

$$F = Q - \frac{\overline{Q}}{rst} + Q\overline{Q}\frac{(1-r)(1-s)(1-t)}{rst} = \sum_{i,j,k} c_{ijk} r^i s^j t^k.$$

Definition

The *equivariant vertex measure* is obtained by "swapping the roles of addition and multiplication" in F:

$$\mathbf{w}(\pi) = \prod_{i,j,k} (iu + jv + kx)^{-c_{ijk}}.$$

We use the variables u, v, and x instead of r, s, and t post-swap.

Motivation

Maulik, Nekrasov, Okounkov and Parharipande give a generating function for $\mathbf{w}(\pi)$ in their 2005 paper.

Theorem (MNOP, 2005)

$$Z := \sum_{\pi} \mathbf{w}(\pi) q^{|\pi|} = M(q)^{-\frac{(u+v)(v+x)(x+u)}{uvx}}$$

Example (in lieu of proof. . .)

Consider the unique plane partition π of size 1.
Only the $i = 1$ term of $M(q)$ yields any q^1 terms:

$$[q^1](1-q)^{-\frac{(u+v)(v+x)(x+u)}{uvx}} = \frac{(u+v)(v+x)(x+u)}{uvx}.$$

$$\mathbf{w}(\pi) = (-v-x)(-u-x)(-u-v)(-x)^{-1}(-v)^{-1}(-u)^{-1}$$
$$= \frac{(v+x)(u+x)(u+v)}{uvx}.$$

Motivation

The proof of Z is geometric. One could hope for a combinatorial proof; however, that is currently out of reach.

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The subject of this talk is a warm-up problem for this: the same problem one dimension down.

In two dimensions. . .

Definitions

$$Q_2(\lambda) = \sum_{(i,j)\in\lambda} r^i s^j$$

$$\overline{Q}_2(\lambda) = \sum_{(i,j)\in\lambda} r^i s^j$$

$$F_2(\lambda) = F_2 = -Q_2 - \frac{\overline{Q}_2}{rs} + Q_2\overline{Q}_2\frac{(1-r)(1-s)}{rs} = \sum_{i,j} c_{ij} r^i s^j$$

In two dimensions...

Example

Note that Q_2 assigns a monomial to each box in a shape λ which describes the (matrix) coordinates of the box.

1	r	r^2
s	rs	

Next, we define an operation on Laurent polynomials which switches the roles of addition and multiplication.

Definition

Let $G = \sum_{i,j} d_{i,j} r^i s^j$ be a Laurent polynomial in the variables r and s with no constant term. Then define the *swap* of G to be

$$\mathrm{swap}(G) = \prod_{i,j} (iu - jv)^{d_{i,j}}.$$

Things to note: sign convention, variable changes

Definition

The *equivariant edge measure* is

$$w_{\mathrm{MNOP}}(\lambda) := \mathrm{swap}(F_2(\lambda)) = \prod_{i,j} (iu - jv)^{c_{ij}}.$$

So what is w_{MNOP}?

Reminder: Our goal is to give some combinatorial meaning to w_{MNOP}.

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The notion of w_{MNOP} comes from areas of algebraic geometry (specifically, Hilbert schemes and Donaldson-Thomas theory) in which Jack polynomials frequently arise. However, this particular connection appears to be new.

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In order to talk about the Jack Plancherel measure, let's first define the (ordinary) Plancherel measure.

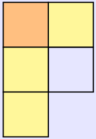
Hook Lengths

Definition

Given a cell (i, j) (in matrix coordinates) in a Young diagram λ, the *hook length* of the cell (i, j) is

$$h((i,j)) = 1 + (\lambda_i - j) + (\lambda'_j - i).$$

Example



The hook length of the top left box is four.

Plancherel Measure

Theorem (Frame-Robinson-Thrall)

The number of standard Young tableaux of shape λ is

$$f^\lambda = \frac{n!}{\prod_{\square \in \lambda} h(\square)}$$

where $h(\square)$ is the hook length of $\square \in \lambda$ and $|\lambda| = n$.

Theorem (Young-Frobenius)

For any integer $n > 0$,

$$1 = \sum_{\lambda \vdash n} \frac{(f^\lambda)^2}{n!}.$$

Plancherel Measure

Combining these two theorems, we obtain a probability measure on Young diagrams.

Theorem (Plancherel Measure)

We have
$$1 = \sum_{\lambda \vdash n} \frac{n!}{\left(\prod_{\square \in \lambda} h(\square)\right)^2}.$$

Jack Plancherel Measure

We can do all of this in a Jack setting as well.

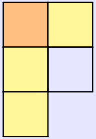
Definition

The *upper* and *lower hook lengths* of a cell (i,j) in a Young diagram λ are

$$h^*((i,j)) = u + u(\lambda_i - j) + v(\lambda'_j - i)$$
$$h_*((i,j)) = v + u(\lambda_i - j) + v(\lambda'_j - i)$$

where u and v are the (homogenized) Jack parameters.

Example



$$h^*((0,0)) = 2u + 2v$$
$$h_*((0,0)) = u + 3v$$

Jack Plancherel Measure

Theorem (Jack Plancherel Measure)

We have
$$1 = \sum_{\lambda \vdash n} \frac{n!(uv)^n}{\prod_{\square \in \lambda} h^*(\square) h_*(\square)}.$$

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$$1 = \sum_{\lambda \vdash n} \frac{n!(uv)^n}{\prod_{\square \in \lambda} h^*(\square) h_*(\square)}.$$

For our result, we need a slightly different version of this.

Definition

Define

$$w_{\text{Jack}}(\lambda) = \frac{1}{\prod_{\square \in \lambda} h^*(\square) h_*(\square)}.$$

Main Result

Theorem (P.-Young)

We have

$$w_{Jack} = -w_{MNOP}.$$

Proof Idea

It turns out that

$$F_2(\lambda) = F_2 = -Q_2 - \frac{\overline{Q}_2}{rs} + Q_2\overline{Q}_2\frac{(1-r)(1-s)}{rs}$$

is difficult to work with because of the last term. Instead, our proof is inductive, starting with the one-box shape and adding one box at a time.

Proof Idea

In other words, we show that

$$\frac{\mathrm{swap}(F_2(\lambda))}{\mathrm{swap}(F_2(\mu))} = \frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)}.$$

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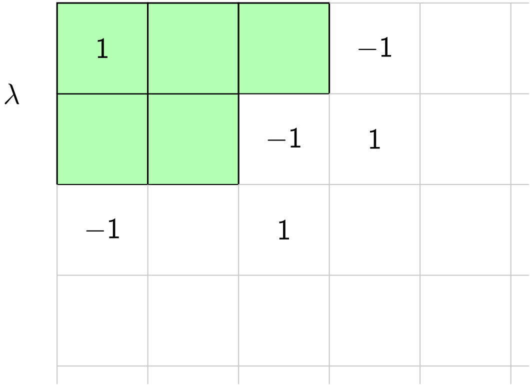
This allows us to avoid the messiness of $F_2(\lambda)$ because

$$\frac{\mathsf{swap}(F_2(\lambda))}{\mathsf{swap}(F_2(\mu))} = \mathsf{swap}(F_2(\lambda) - F_2(\mu))$$

and $F_2(\lambda) - F_2(\mu)$ is much cleaner.

The Corner Polynomial

One reason that this is cleaner is because the "corner polynomial" shows up.



The Corner Polynomial

Lemma (P.-Young)

The "corner polynomial" for a partition λ is

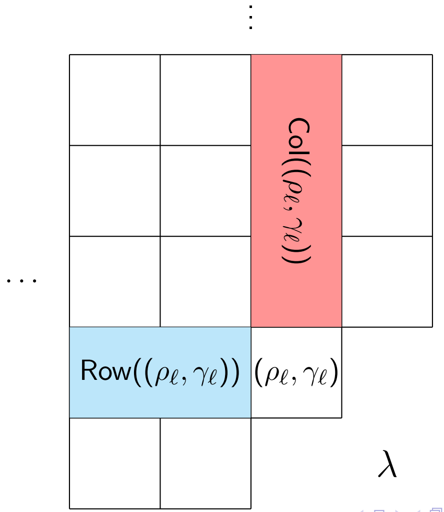
$$C = C(\lambda) := Q_2(1-r)(1-s)$$

$$= 1 + \sum_{(i,j) \text{ inside corner of } \lambda} r^{i+1}s^{j+1} - \sum_{(i,j) \text{ outside corner of } \lambda} r^i s^j$$

$$= 1 + \sum_{k=1}^{m} r^{\rho_k+1}s^{\gamma_k+1} - \sum_{k=1}^{m+1} r^{\rho_k+1}s^{\gamma_{k-1}+1}.$$

Proof Idea (continued)

It turns out that a lot of cancellation occurs in $\frac{w_{\text{Jack}}(\lambda)}{w_{\text{Jack}}(\mu)}$, so all that is left is a product over the boxes directly to the left and above the added box λ/μ.



Proof Idea (continued)

After these simplifications, one can show that

$$\frac{w_{\mathsf{MNOP}}(\lambda)}{w_{\mathsf{MNOP}}(\mu)} = \frac{w_{\mathsf{Jack}}(\lambda)}{w_{\mathsf{Jack}}(\mu)}.$$

(However, it is tedious.)

Proof Idea (continued)

After these simplifications, one can show that

$$\frac{w_{\mathrm{MNOP}}(\lambda)}{w_{\mathrm{MNOP}}(\mu)} = \frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)}.$$

(However, it is tedious.) Routine induction yields our theorem.

Theorem (P.-Young)

We have

$$w_{Jack} = -w_{MNOP}.$$

In other words, the equivariant edge measure of MNOP is the Jack Plancherel measure (up to conventions).

Back to Motivations

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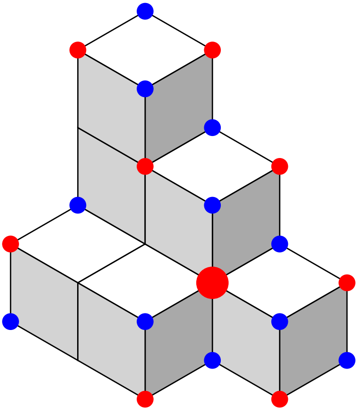
It's much more difficult, but we now know that $\mathbf{w}(\pi)$ is an analogue for plane partitions of the Jack Plancherel measure.

After an initial computer experiment, it appears that (an analogue of) the corner polynomial shows up in the same way.



Thank you!