Jack Combinatorics of the Equivariant Edge Measure

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October 18, 2024

Motivation

Definition (Stanley)

A *plane partition* is an array $\pi = (\pi_{i,j})_{i,j \geq 1}$ of nonnegative integers such that π has finite support (i.e. finitely many nonzero entries) and is weakly decreasing in the rows and columns.

Example

3	2	2
1	1	

Motivation

Definition

The sum of all of the entries in a plane partition π is the *size* of π. We denote this $|\pi|$.

Theorem (MacMahon)

The number of plane partitions with size n is the coefficient of q^n in

$$M(q) = \prod_{i \geq 1} \left(\frac{1}{1 - q^i} \right)^i .$$

Motivation

Definition

Define

$$Q(\pi) = \sum_{(i,j,k)\in\pi} r^i s^j t^k$$

$$\overline{Q}(\pi) = \sum_{(i,j,k)\in\pi} r^{-i} s^{-j} t^{-k}.$$

Example

3	2	2
1	1	

Given the plane partition π as before,

$$Q = 1 + r + r^2 + s + rs + t + rt + r^2 t$$

$$\overline{Q} = 1 + r^{-1} + r^{-2} + s^{-1} + r^{-1}s^{-1} + t^{-1} + r^{-1}t^{-1} + r^{-2}t^{-1}.$$

Motivation

Definition

From Q and $\overline{Q}$ define

$$F = Q - \frac{\overline{Q}}{rst} + Q\overline{Q}\frac{(1-r)(1-s)(1-t)}{rst} = \sum_{i,j,k} c_{ijk} r^i s^j t^k.$$

Definition

The *equivariant vertex measure* is obtained by "swapping the roles of addition and multiplication" in F:

$$\mathrm{w}(\pi) = \prod_{i,j,k} (iu + jv + kw)^{-c_{ijk}}.$$

We use the variables u, v, and w instead of r, s, and t post-swap.

Motivation

Maulik, Nekrasov, Okounkov and Parharipande give a generating function for $\mathrm{w}(\pi)$ in their 2005 paper.

Theorem (MNOP, 2005)

$$Z := \sum_{\pi} \mathrm{w}(\pi) q^{|\pi|} = M(q)^{-\frac{(u+v)(v+w)(w+u)}{uvw}}$$

Example (in lieu of proof...)

Consider the unique plane partition π of size 1:

$$\begin{aligned}\mathrm{w}(\pi) = (-v-w)(-u-w)(-u-v)(-w)^{-1}(-v)^{-1}(-u)^{-1} \\ = \frac{(v+w)(u+w)(u+v)}{uvw}.\end{aligned}$$

Only the $i = 1$ term of $M(q)$ yields any q^1 terms:

$$[q^1](1-q)^{\frac{(u+v)(v+w)(w+u)}{uvw}} = \frac{(u+v)(v+w)(w+u)}{uvw}.$$

Motivation

The proof of Z is geometric and one could hope for a combinatorial proof; however, that is currently out of reach.

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The subject of this talk is a warm-up problem for this: the same problem one dimension down.

Hook Lengths

Definition

Given a cell (i, j) (in matrix coordinates) in a Young diagram λ, the *hook length* of the cell (i, j) is

$$h((i,j)) = 1 + (\lambda_i - j) + (\lambda'_j - i).$$

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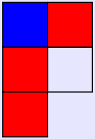
Example



The hook length of the blue box is four.

Plancherel Measure

Theorem (Frame-Robinson-Thrall)

The number of standard Young tableaux of shape λ is

$$f^\lambda = \frac{n!}{\prod_{\square \in \lambda} h(\square)}$$

where $h(\square)$ is the hook length of $\square \in \lambda$.

Theorem (Young-Frobenius)

For any integer $n > 0$,

$$1 = \sum_{\lambda \vdash n} \frac{(f^\lambda)^2}{n!}.$$

Motivation

Combining these two theorems, we obtain a probability measure on standard Young tableaux.

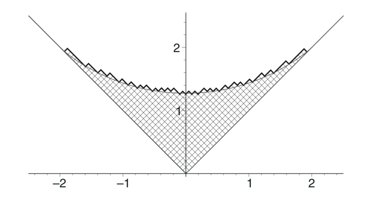
Theorem (Plancherel Measure)

We have

$$1 = \sum_{\lambda \vdash n} \frac{n!}{\left(\prod_{\square \in \lambda} h(\square)\right)^2}.$$

A theorem of Kerov shows that generating large Plancherel-random tableau yields a limit shape.

Example



This image was taken from a paper of Okounkov.

Jack Plancherel Measure

We can do all of this in a Jack setting as well.

Definition

The *upper* and *lower hook lengths* of a cell (i, j) in a Young diagram λ are

$$h^*((i,j)) = t + t(\lambda_i - j) + (\lambda'_j - i)$$
$$h_*((i,j)) = 1 + t(\lambda_i - j) + (\lambda'_j - i)$$

where t is the Jack parameter.

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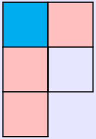
Example



$$h^*((0,0)) = 2t + 2$$
$$h_*((0,0)) = t + 3$$

Jack Plancherel Measure

Theorem (Jack Plancherel Measure)

We have

$$1 = \sum_{\lambda \vdash n} \frac{n!t^n}{\prod_{\square \in \lambda} h^*(\square) h_*(\square)}.$$

Just as before, if we generate a large random Young diagram from this, we'll get a limit shape. (Dołega) For our result, we need a slightly different version of this.

Definition

Define

$$w_{\text{Jack}}(\lambda) = \frac{1}{\prod_{\square \in \lambda} h^*(\square) h_*(\square)}.$$

Main Result

Theorem (P.-Young)

We have

$$w_{Jack} = -w_{MNOP}.$$

Okay, but what does the right side mean?

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Okay, but what does the right side mean?

This is the two-dimensional version of w.

Definition

$$Q_2(\lambda) = \sum_{(i,j)\in\lambda} r^i s^j$$

$$\overline{Q}_2(\lambda) = \sum_{(i,j)\in\lambda} r^i s^j$$

$$F_2(\lambda) = F_2 = Q_2 - \frac{\overline{Q}_2}{rs} + Q_2\overline{Q}_2\frac{(1-r)(1-s)}{rs} = \sum_{i,j} c_{ij} r^i s^j$$

Main Result

Example

Note that Q assigns a monomial to each box in a shape λ which describes the (matrix) coordinates of the box.

1	r	r^2
s	rs	

Next, we define an operation on Laurent polynomials which switches the roles of addition and multiplication.

Definition

Let $G = \sum_{i,j} d_{i,j} r^i s^j$ be a Laurent polynomial in the variables r and s with no constant term. Then define the *swap* of G to be

$$\text{swap}(G) = \prod_{i,j} (iu - jv)^{-d_{i,j}}.$$

Definition

We have

$$w_{\text{MNOP}}(\lambda) = \text{swap}(F_2(\lambda)).$$

Main Result

Definition

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$$w_{\mathrm{MNOP}}(\lambda) = \mathrm{swap}(F_2(\lambda)).$$

The notion of w_{MNOP} comes from algebraic geometry (specifically, Hilbert schemes and Donaldson-Thomas theory) in which Jack polynomials do frequently arise. However, this particular connection appears to be new.

Proof Idea

It turns out that

$$F_2(\lambda) = F_2 = Q_2 - \frac{\overline{Q}_2}{rs} + Q_2\overline{Q}_2\frac{(1-r)(1-s)}{rs}$$

is difficult to work with because of the last term. Instead, our proof is inductive, starting with the one-box shape and adding one box at a time.

Proof Idea

In other words, we need to show that

$$\frac{w_{\mathrm{MNOP}}(\lambda)}{w_{\mathrm{MNOP}}(\mu)} := \frac{\mathrm{swap}(F_2(\lambda))}{\mathrm{swap}(F_2(\mu))} = \frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)}.$$

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This allows us to avoid the messiness of $F_2(\lambda)$ because

$$\frac{\text{swap}(F_2(\lambda))}{\text{swap}(F_2(\mu))} = \text{swap}((F_2(\lambda)) - F_2(\mu))$$

and $F_2(\lambda)) - F_2(\mu)$ is much cleaner.

The Corner Polynomial

One reason that this is cleaner is because the "corner polynomial" shows up.

Lemma (P.-Young)

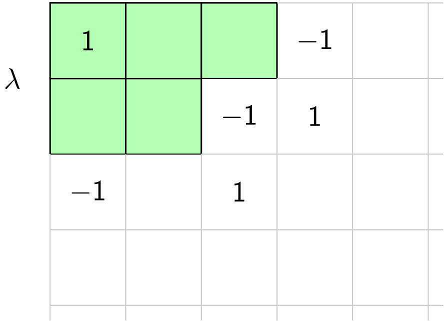
The "corner polynomial" for a partition λ is

$$C = C(\lambda) := Q_2(1-r)(1-s)$$

$$= 1 + \sum_{(i,j) \text{ inside corner of } \lambda} r^{i+1}s^{j+1} - \sum_{(i,j) \text{ outside corner of } \lambda} r^i s^j$$

$$= 1 + \sum_{k=1}^{m} r^{\rho_k+1}s^{\gamma_k+1} - \sum_{k=1}^{m+1} r^{\rho_k+1}s^{\gamma_{k-1}+1}.$$

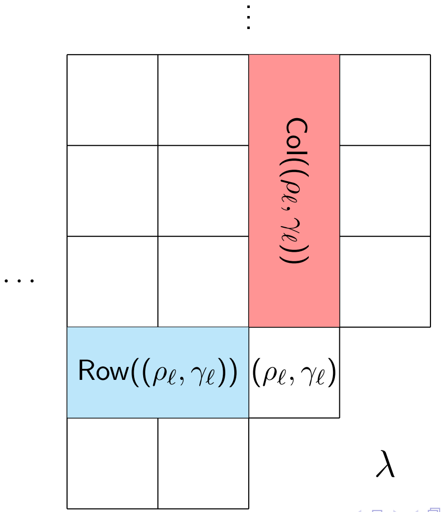
The Corner Polynomial



Inside every cell in $\lambda = (3, 2)$ is the coefficient of its contribution to C. Empty cells contribute nothing to C. For example, the cell $(1, 2)$ contributes $-1 \cdot r^1 s^2$ to C.

Proof Idea (continued)

It turns out that a lot of cancellation occurs in $\frac{w_{\mathsf{Jack}}(\lambda)}{w_{\mathsf{Jack}}(\mu)}$, so all that is left is a product over the boxes directly to the left and above the added box λ/μ.



Proof Idea

After these simplifications, we were able to show that

$$\frac{w_{\mathrm{MNOP}}(\lambda)}{w_{\mathrm{MNOP}}(\mu)} = \frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)}.$$

(However, it was tedious.)

Proof Idea

After these simplifications, we were able to show that

$$\frac{w_{\mathrm{MNOP}}(\lambda)}{w_{\mathrm{MNOP}}(\mu)} = \frac{w_{\mathrm{Jack}}(\lambda)}{w_{\mathrm{Jack}}(\mu)}.$$

(However, it was tedious.) Routine induction yields our theorem.

Theorem (P.-Young)

We have

$$w_{Jack} = -w_{MNOP}.$$

In other words, the equivariant *vertex* measure of MNOP is the Jack Plancherel measure (up to conventions).

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It's much more difficult, but we now know that w(π) is an analogue for plane partitions of the Jack Plancherel measure.

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After an initial computer experiment done by Ben, it appears that (an analogue of) the corner polynomial shows up in the same way and seems related to cluster algebras.

So Ben and I are working with Kayla this term to see what we can find.

Thank you!