# Weeks 1 and 2 Lecture Notes

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## Section 2.1: A Preview of Calculus

- Let's consider the following situation: you are driving from Eugene to Portland. You get on the highway at mile marker 189 at 2 PM and get off the highway at mile marker 292 at 3:25 PM.
  - What is your average speed throughout the trip?
  - What does your speedometer read exactly one hour into your trip? Can you tell? Why or why not?
  - What if I told you that $t$ minutes after departing, the mile marker you are passing is given by the function $x(t) = 1.21t + 189$? Can you now tell what your speedometer reads exactly one hour into your trip?
  - What if I told you that $t$ minutes after departing, the mile marker you are passing is given by the function $x(t) = .014t^2 + 189$? Can you now tell what your speedometer reads exactly one hour into your trip?
- The point of the previous questions is to find out how great lines are.
- Knowing that speed is an average rate of change and that linear functions have constant rate of change, we can recover your instantaneous speed from your position function: it's the slope!
- But for nonlinear functions, we don't have this nice slope number around to tell us about your speed.
- That's not to say that you don't have a speed, though.
- If you were to take a picture of your speedometer one hour after you left in part (d), it would still read something. And we should be able to figure it out based on the function $x(t)$.
- So how do we do that?
- First, maybe we would figure out the average speed from 60 minutes to 61 minutes.
- Then, maybe we would figure out the average speed from 60 minutes to 60.1 minutes. Then maybe from 60 to 60.01, etc. and see if the numbers that we get are getting closer to anything...
- These average speeds that we're computing can be thought of as slopes (draw picture)
- Each of these lines that we're drawing, which passes through two distinct points on the graph of $x(t)$ is called a secant line.
- **Def:** For a function, $x(t)$ and $t$-values $t_0$ and $t_1$ in the domain of $x(t)$, the secant line to $x(t)$ through $t = t_0$ and $t = t_1$ is the line which passes through $(t_0, x(t_0))$ and $(t_1, x(t_1))$. It has slope

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0}$$

- **Ex:** The line we drew through the points $(60, x(60))$ and $(61, x(61))$ was the secant line to $x(t)$ through $t = 60$ and $t = 61$.
- We'll come back to this example later after we have a tool under our belts called the limit.
- **Ex:** Find the equation of the secant line to the function $f(x) = x^2 + 1$ through...
  - ...$x = 3$ and $x = 4$
  - ...$x = 3$ and $x = 3.1$
  - ...$x = 3$ and $x = 3.01$

## Section 2.2: The Limit of a Function

- Let's look at the graphs of three different functions:
  - $f(x) = \frac{x^2-4}{x-2}$
  - $g(x) = \frac{|x-2|}{x-2}$
  - $h(x) = \frac{1}{(x-2)^2}$
- What's similar about their behavior at $x = 2$?
- What's different about their behavior near $x = 2$?
- The functions' behaviors at $x = 2$ are questions that you could have answered in 111 and 112, so we won't go into that in this class.
- The second question, however, is a new question, and it's what we're trying to answer with limits. As our $x$-values get closer and closer to 2, what happens to the output values?
- For $f(x)$, the output values get closer and closer to 4.
- For $g(x)$, the output values get close to $-1$ from the left, but $+1$ from the right
- For $h(x)$, the output values grow without bound and get "closer and closer" to $\infty$.
- This brings us to our working definition of a limit:
- **Def:** Let $f(x)$ be a function defined at all values in an open interval containing $a$, with the possible exception of $a$ itself. Let $L$ be a real number. If all values of the function $f(x)$ approach the real number $L$ as the values of $x$ approach the number $a$, then we say that the "limit of $f(x)$ as $x$ approaches $a$ is $L$." We abbreviate this to $\lim_{x\to a} f(x) = L$.
- **Ex:** $\lim_{x\to 2} f(x) = 4$
- **Ex:** $\lim_{x\to 2} g(x)$ does not exist (or DNE). Note: don't write "= DNE"
- **Ex:** $\lim_{x\to 2} h(x)$ as we've defined it, DNE. But you can guess what we'll put in later.
- A strategy for guessing limits: table of values
- **Ex:** Guess the value of $\lim_{x\to 0} \frac{\sin(x)}{x}$ using a table of values (make sure your calculator is set to radians!).

| $x$ | $\frac{\sin(x)}{x}$ | | $x$ | $\frac{\sin(x)}{x}$ |
|---|---|---|---|---|
| -0.1 | 0.9983 | | 0.1 | 0.9983 |
| -0.01 | 0.999983 | | 0.01 | 0.999983 |
| -0.001 | 0.99999983 | | 0.001 | 0.99999983 |

- **Ex:** Guess the value of $\lim_{x\to -1} \frac{x^5+x^4+x+1}{x+1}$ using a table of values.

| $x$ | $\frac{x^5+x^4+x+1}{x+1}$ | | $x$ | $\frac{x^5+x^4+x+1}{x+1}$ |
|---|---|---|---|---|
| -1.1 | 2.464 | | -.9 | 1.6561 |
| -1.01 | 2.0406 | | -.99 | 1.961 |
| -1.001 | 2.004 | | -.999 | 1.996 |

- A second strategy for guessing limits is with a graph:
- **Ex:** Consider the function

$$j(x) = \begin{cases} -x & x < -1 \\ -2 & x = -1 \\ 1 & -1 < x < 2 \\ \sqrt{x+2} & x > 2 \end{cases}$$

  - Draw the graph of $j(x)$
  - Compute $\lim_{x\to -1} j(x)$
  - Compute $\lim_{x\to 2} j(x)$
  - Compute $\lim_{x\to 0} j(x)$
- Important observations so far:
  - The limit can exist, even if the function doesn't
  - The function can exist, even if the limit doesn't
  - Sometimes, neither the function nor the limit exist
  - Sometimes, both the function and the limit exist and they are different
  - Sometimes, both the function and the limit exist and they are the same.
- The lesson so far: computing the limit is different from computing the function value.
- So far, the only times we've seen limits fail to exist is when we have different behavior from either side of the point in question.
- Maybe a helpful tool, then, is to talk about limits, but instead of thinking about all $x$-values near a point, let's think about all $x$-values less than a point, or all $x$-values bigger than a point.
- For instance, going back to our function $g(x) = \frac{|x-2|}{x-2}$, if we only look at $x$-values to the left of 2, the $y$-values get closer and closer to $-1$. If we only look at $x$-values to the right, the $y$-values get closer and closer to 1.
- **Def'n:** Let $f(x)$ be a function defined on all values in an open interval of the form $(b, a)$ and let $L$ be a real number. If the values of $f(x)$ approach $L$ as the values of $x$ approach $a$ from within the interval $(b, a)$, then we say that the limit of $f(x)$ as $x$ approaches $a$ from the left is $L$. Notation: $\lim_{x\to a^-} f(x) = L$
- **Def'n:** Let $f(x)$ be a function defined on all values in an open interval of the form $(a, c)$ and let $L$ be a real number. If the values of $f(x)$ approach $L$ as the values of $x$ approach $a$ from within the interval $(a, c)$, then we say that the limit of $f(x)$ as $x$ approaches $a$ from the right is $L$. Notation: $\lim_{x\to a^+} f(x) = L$
- **Ex:** For our function

$$j(x) = \begin{cases} -x & x < -1 \\ -2 & x = -1 \\ 1 & -1 < x < 2 \\ \sqrt{x+2} & x > 2 \end{cases}$$

  compute $\lim_{x\to 2^-} j(x)$ and $\lim_{x\to 2^+} j(x)$

- The left-hand limit looks at $y$-values as $x$ approaches the point from the left. The right-hand limit looks at $y$-values as $x$ approaches the point from the right. The general limit looks at $y$-values as $x$ approaches the point from both sides. You'll sometimes see the general limit called the two-sided limit as a result.
- So far, every function that we've seen has no problems with its one-sided limits. Is it the case that every function has one-sided limits everywhere?
- No! Consider the function $f(x) = \sin\left(\frac{1}{x}\right)$ near $x = 0$. Let's look at a table of values for this function:

| $x$ | $\sin\left(\frac{1}{x}\right)$ | | $x$ | $\sin\left(\frac{1}{x}\right)$ |
|---|---|---|---|---|
| $-0.1$ | $0.544$ | | $0.1$ | $-0.544$ |
| $-0.01$ | $0.506$ | | $0.01$ | $-0.506$ |
| $-0.001$ | $-0.827$ | | $0.001$ | $0.827$ |
| $-0.0001$ | $0.306$ | | $0.0001$ | $-0.306$ |
| $-0.00001$ | $-0.0357$ | | $0.00001$ | $0.0357$ |

- $\lim_{x\to 0^-} \sin\left(\frac{1}{x}\right)$ DNE and $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ DNE
- In what other ways can limits be weird?
- Remember our function $h(x) = \frac{1}{(x-2)^2}$ from earlier?
- Our current definitions of limits only let us say that $\lim_{x\to a} f(x) = L$ if $L$ is a real number. But the graph makes it look a lot like we want to write $\lim_{x\to 2} h(x) = \infty$, despite the fact that $\infty$ is not a number.
- So we're going to let ourselves write that:
- **Def'n:** Let $f(x)$ be a function defined on an open interval $(b, c)$ (except possibly at $a$) where $b < a < c$. Then
  1. If the values of $f(x)$ increase without bound as $x$ approaches $a$ from within the interval $(b, a)$, we write $\lim_{x\to a^-} f(x) = \infty$
  2. If the values of $f(x)$ decrease without bound as $x$ approaches $a$ from within the interval $(b, a)$, we write $\lim_{x\to a^-} f(x) = -\infty$
  3. If the values of $f(x)$ increase without bound as $x$ approaches $a$ from within the interval $(a, c)$, we write $\lim_{x\to a^+} f(x) = \infty$
  4. If the values of $f(x)$ decrease without bound as $x$ approaches $a$ from within the interval $(a, c)$, we write $\lim_{x\to a^+} f(x) = -\infty$
  5. If $\lim_{x\to a^-} f(x) = \pm\infty = \lim_{x\to a^+} f(x)$, then we write $\lim_{x\to a} f(x) = \pm\infty$
- **Ex:** Use a table of values to compute
  1. $\lim_{x\to 0^-} \frac{1}{x}$
  2. $\lim_{x\to 0^+} \frac{1}{x}$
  3. $\lim_{x\to 0} \frac{1}{x}$

| $x$ | $\frac{1}{x}$ | | $x$ | $\frac{1}{x}$ |
|---|---|---|---|---|
| $-1$ | $-1$ | | $1$ | $1$ |
| $-0.1$ | $-10$ | | $0.1$ | $10$ |
| $-0.01$ | $-100$ | | $0.01$ | $100$ |
| $-0.001$ | $-1000$ | | $0.001$ | $1000$ |

- **Ex:** Compute
  1. $\lim_{x\to 5^-} \frac{10}{(x-5)^4}$

2. $\lim_{x\to 5^+} \frac{10}{(x-5)^4}$
3. $\lim_{x\to 5} \frac{10}{(x-5)^4}$

- This limit language allows us to give a good definition of a term that you're probably already familiar with: asymptotes.
- **Def'n:** Let $f(x)$ be a function. If $\lim_{x\to a^-} f(x) = \pm\infty$ and $\lim_{x\to a^+} f(x) = \pm\infty$, then the line $x = a$ is a vertical asymptote for $f(x)$.
- **Ex:** $f(x) = \frac{1}{x}$ has a vertical asymptote at $x = 0$.
- **Ex:** $f(x) = \frac{1}{x+10}$ has a vertical asymptote at $x = -10$.
- **Ex:** $f(x) = \frac{x^2-4}{x-2}$ has no vertical asymptotes (we had the graph of this one earlier).

## Section 2.3: The Limit Laws

- So far, all we've been able to do is estimate limits using tables and graphs.
- It would be great to have some rigorous rules for computing limits
- Let's start with some simple ones: For any real numbers $a$ and $c$
  - $\lim\limits_{x\to a} c = c$
  - $\lim\limits_{x\to a} x = a$
- Justify using graphs
- More generally (and more usefully), we have the following limit laws: Let $f(x)$ and $g(x)$ be defined for all $x$ in some interval containing $a$, with the possible exception of $a$ itself. Suppose that $L$ and $M$ are real numbers so that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Let $c$ be a constant. Then each of the following holds:
  - Sum law: $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$
  - Product law: $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)] = L \cdot M$
  - Power law: $\lim_{x\to a} f(x)^c = [\lim_{x\to a} f(x)]^c = L^c$ whenever $L^c$ is a well-defined, real number.
- The book lists a number of other laws, which aren't wrong, but these are the only three that you really need.
- Justify the limit laws, vaguely.
- Let's do some problems using them.
- **Ex:** $\lim_{x\to 3} x + 5$
- **Ex:** $\lim_{t\to -1} 2t^3$
- **Ex:** $\lim_{t\to 5} \frac{2t}{\sqrt{10t-4}}$
- Okay, even if we can do a bunch of limits using the rules that we have, it's helpful if we state a few more rules. In the same situation as the previous laws...
  - Difference law: $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$
  - Constant multiple law: $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL$
  - Quotient law: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ if $M \neq 0$

- **Ex:** $\lim_{h\to -2} \frac{3h^4-5h}{h+1}$
- So by now you've probably noticed that we're just plugging in the number to the variable at this point.
- For a lot of our functions, we're noticing that $\lim_{x\to a} f(x) = f(a)$. This is not always the case however.
- Here are some cases when it works:
- **Thm:** Let $p(x)$ and $q(x)$ be polynomials and let $a$ be a real number. Then $\lim_{x\to a} p(x) = p(a)$. Additionally, $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ when $q(a) \neq 0$.
- But what if $q(a) = 0$? We don't want to forget that case. Let's go back to our tables of values to get some intuition.
- Create a table of values to estimate the following limits
  1. $\lim_{x\to 2} \frac{x+3}{(x-2)^2}$
  2. $\lim_{x\to 2} \frac{x^2-4}{x-2}$
  3. $\lim_{x\to 2} \frac{x^2-4x+4}{x-2}$

| $x$ | $\frac{x+3}{(x-2)^2}$ | $\frac{x^2-4}{x-2}$ | $\frac{x^2-4x+4}{x-2}$ | | $x$ | $\frac{x+3}{(x-2)^2}$ | $\frac{x^2-4}{x-2}$ | $\frac{x^2-4x+4}{x-2}$ |
|---|---|---|---|---|---|---|---|---|
| 1.9 | 490 | 3.9 | $-.1$ | | 2.1 | 510 | 4.1 | .1 |
| 1.99 | 49900 | 3.99 | $-.01$ | | 2.01 | 50100 | 4.01 | .01 |
| 1.999 | 4999000 | 3.999 | $-.001$ | | 2.001 | 5001000 | 4.001 | .001 |

- Let's analyze these three limits
- In the first one, the numerator gets close to 5 while the denominator gets close to 0.
- This makes the whole fraction blow up—the fraction is positive for $x$ a little bit bigger than 2 and for $x$ a little bit smaller than 2, so the whole fraction goes to $\infty$
- In the second case, both the numerator and the denominator get close to 0.
- We've seen the second example before. This was our $f(x)$ in 2.2
- In 2.2 we used a graph.
- But the way we graphed the function was first by simplifying the arithmetic.
- We observed that $x + 2$ and $\frac{x^2-4}{x-2}$ have the same behavior near $x = 2$, so they have the same limit.
- Thus, $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} x + 2 = 4$
- Likewise, for the third example, the numerator and denominator go to 0. Using a similar method as the previous problem, we find $\lim_{x\to 2} \frac{x^2-4x+4}{x-2} = \lim_{x\to 2} \frac{(x-2)^2}{x-2} = \lim_{x\to 2} x - 2 = 0$
- We should create a decision making chart when $p(x)$ and $q(x)$ are polynomials. [Draw flowchart.]
- **Ex:** $\lim_{x\to 5} \frac{2x}{x^3+1}$
- **Ex:** $\lim_{x\to 5} \frac{x^2-4x-5}{10-2x}$
- **Ex:** $\lim_{x\to 5} \frac{x-5}{(x^2-10x+25)(x+2)}$
- Some other hard things that can happen:
- **Ex:** $\lim_{x\to 3} \frac{x-3}{\sqrt{x+1}-2}$

- **Ex:** $\lim_{x\to 0} \frac{1}{x} + \frac{5}{x(x-5)}$
- General principle: whenever there's a hard thing, eliminate the hard thing first. Then deal with the remainder of the problem
- Furthermore, it turns out that piecewise functions are okay with limits, provided the pieces are okay with limits
- **Ex:** Let $f(x) = \begin{cases} 2x-1 & x < -1 \\ (x+3)^4 & x \geq -1 \end{cases}$ Compute
  1. $\lim_{x\to -1^-} f(x)$
  2. $\lim_{x\to -1^+} f(x)$
  3. $\lim_{x\to -1} f(x)$
  4. $\lim_{x\to -4} f(x)$
- We can do a lot of limits now. But things that we don't have limit laws for are trig functions, exponentials, and logs
- One of the most important tools that we have for dealing with trig functions is the squeeze theorem:
- **Thm:** Let $f(x), g(x), h(x)$ be defined in an open interval containing $a$ (except possibly at $a$) and let $L$ be a real number. If for all $x \neq a$, $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} f(x) = L$.
- The squeeze theorem is often tricky for calculus students to apply because it's different than other theorems we've encountered.
- Rather than giving you a formulaic way to do a computation, you need to be creative.
- The limit that you often want to compute is $\lim_{x\to a} f(x)$.
- $g(x)$ and $h(x)$ are not given to you: you need to be creative and come up with good functions on your own.
- **Ex:** $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$
- Note that this is not a limit we can evaluate using our limit laws!

## Section 2.4: Continuity

- By the end of this section, we'll have enough machinery to discuss the following claim: if you were five feet tall last year and you are seven feet tall this year, at some point between last year and this year, you were six feet tall.
- You probably consider this a blindingly obvious claim. You're right.
- But why is it true, really?
- What's the important feature of height that makes that such a blindingly obvious claim?
- Let's vary the claim up a little bit and see what important features the hypotheses (parts between the "if" and the "then") and conclusion (the part after the "then") have that make the argument good.
- Variant 1: If you were five feet tall last year and you are seven feet tall this year, at some point between last year and this year you were eight feet tall.
- Now we see that we need to pick an <u>intermediate</u> height

- Variant 2: If you were five feet tall last year and you are seven feet tall this year, at some point next year, you'll be six feet tall
- Now we see that the conclusion needs to involve an intermediate time
- Variant 3: If you had exactly five dollars last year and you have exactly seven dollars now, at some point between last year and this year, you had exactly six dollars.
- Now we see that the quantity we are considering needs to not have gaps!
- But what does it mean to not have gaps?
- Maybe one way of saying a quantity has no gaps is if you draw its graph, you don't have to pick up your pencil when drawing.
- You can draw it in one continuous stroke.
- Continuous is the word we want to use to describe functions that don't have jumps.
- But how should we define it?
- We want to think first about continuity at a point. In what ways might you have to pick up your pencil when drawing the graph of a function through a point?
- One thing that might occur is if the function isn't defined at the point in question (draw picture)
- Another thing that might occur is if the limit doesn't exist (draw picture)
- Another thing that might occur is that the limit and the function exist, but they disagree.
- **Def:** Let $a \in \mathbb{R}$ (shorthand for "$a$ is a real number") and let $f(x)$ be a function defined on an interval containing $a$. Then $f(x)$ is continuous at $a$ if $\lim_{x\to a} f(x) = f(a)$.
- As before, we want to develop our intuition for this geometrically before moving on to the algebra.
- **Ex:** Consider the graph of the function $f(x)$. Is it continuous at certain points?
- **Ex:** Make piecewise function continuous.
- Now what about the algebra?
- The definition should look a little familiar to us, since it's saying that a function is continuous if when computing the limit, you can just plug in the limiting value.
- What types of functions do we have where we know we can do this?
- Polynomials! Or rational functions when the denominator is not zero.
- **Thm:** Let $p(x)$ and $q(x)$ be polynomials. Then they are continuous at every real number (i.e. $\lim_{x\to a} p(x) = p(a)$ for all $a \in \mathbb{R}$). Furthermore, $\frac{p(x)}{q(x)}$ is continuous at every $a$ for which $q(a) \neq 0$.
- In some sense, you already know this. If you are drawing the graph of a polynomial, you don't ever have to pick up your pencil. If you are drawing the graph of a rational function, you only have to pick up your pencil when the denominator is zero.
- **Ex:** At which points (if any) is the function $f(x) = x^{150} - 3x^{72} + 4$ discontinuous?
- **Ex:** At which points (if any) is the function $f(x) = \frac{(x-2)(x+1)}{x^2+1}$ discontinuous?
- **Ex:** At which points (if any) is the function $f(x) = \frac{x^2-2x+1}{x^2+2x-3}$ discontinuous?

- **Ex:** At which points (if any) is the function $f(x) = \begin{cases} \frac{x^2-2x+1}{x^2+2x-3} & x \neq -3, 1 \\ 0 & x = -3, 1 \end{cases}$ discontinuous?
- More generally, how do we build continuous functions?
- We have some rules for this: If $f(x)$ and $g(x)$ are continuous at $a$, then
  1. $(f \pm g)(x)$ is continuous at $a$
  2. $(f \cdot g)(x)$ is continuous at $a$
  3. $\left(\frac{f}{g}\right)(x)$ is continuous at $a$ provided $g(a) \neq 0$
  4. $f(x)^c$ is continuous at $a$ provided $x^c$ is defined in an open interval containing $f(a)$.
- You might notice that these rules seem to correspond to our limit laws, and in fact, they do!
- But there's one more operation that we can perform on functions: composition
- Recall that function composition is when you plug one function into another.
- **Ex:** For $f(x) = x^2 - 3x + 2$, what is $f(x+h)$?
- **Ex:** For $f(x) = \sqrt{x}$ and $g(x) = 10x - 15$, what is $(f \circ g)(x)$?
- **Ex:** For $f(x) = \sin(x)$ and $g(x) = x - \pi$, what is $(f \circ g)(x)$?
- Reminder: $\circ$ does not mean multiply!
- Sometimes, for $(f \circ g)(x)$, I refer to $f$ as the outside function and $g$ as the inside function.
- **Thm:** (Composite Function Theorem) Suppose that $a, L \in \mathbb{R}$ with functions $f(x)$ and $g(x)$ so that $g(x)$ is continuous at $a$ and $f(x)$ is continuous at $g(a)$. Then $(f \circ g)(x)$ is continuous at $a$.
- Of particular interest: if $f(x)$ and $g(x)$ are continuous at every point, then so is $(f \circ g)(x)$
- **Fact:** $\sin(x)$ and $\cos(x)$ are continuous at every point.
- **Ex:** At which points (if any) is $\tan(x)$ discontinuous?
- **Ex:** At which points (if any) is $\csc(\pi(x+1))$ discontinuous?
- For this second example, rewrite as $\frac{1}{\sin(\pi(x+1))}$ and note that it's only discontinuous at points where denominator $= 0$.
- One more technical point
- Consider the function $f(x) = \sqrt{x}$
- Is it continuous at 0?
- Well, we want it to be, but it isn't, according to our definition.
- So we add a prepositional phrase to the word "continuous" and we get some new definitions:
- **Def:** A function $f(x)$ defined on an interval $[a, b]$ is <u>continuous from the right</u> at $a$ if $\lim_{x\to a^+} f(x) = f(a)$
- **Def:** A function $f(x)$ defined on an interval $[b, a]$ is <u>continuous from the left</u> at $a$ if $\lim_{x\to a^-} f(x) = f(a)$
- Now $f(x) = \sqrt{x}$ is continuous from the right at $x = 0$.
- We're almost to the point where we can address our height example at the beginning of the section.
- **Def:** Let $a, b \in \mathbb{R}$ with $a < b$. Let $f(x)$ be a function defined on $(a, b)$. Then

1. $f(x)$ is continuous on $(a, b)$ if $f(x)$ is continuous at every point in $(a, b)$
2. $f(x)$ is continuous on $[a, b]$ if $f(x)$ is continuous at every point in $(a, b)$ AND $f(x)$ is continuous from the right at $a$ AND $f(x)$ is continuous from the left at $b$

- **Ex:** Some questions about continuity on intervals.
- Okay, let's go back to our height example.
- How do we relate this to functions? Well, height is a function of time, say $f(x)$
- Rephrasing the statement before (if you were five feet tall a year ago and you are seven feet tall this year, at some point in between, you were six feet tall) into mathematical language, we have
- **Thm:** (Intermediate Value Theorem) Let $f(x)$ be continuous on the interval $[a, b]$. If $z$ is any real number between $f(a)$ and $f(b)$, then there is a $c$ in $[a, b]$ so that $f(c) = z$.
- **Ex:** Show that $f(x) = x - \cos(x)$ has at least one zero.
- Note that we can't actually solve this! But the IVT tells us that somewhere, there is a zero.
- This has interesting philosophical implications (for some people).
- **Ex:** Always true or sometimes false? Let $f(x)$ be a continuous function on $[0, 2]$ with $f(0) > 0$ and $f(2) > 0$. Then $f(x)$ has no zeroes on $[0, 2]$.
- **Ex:** Let $f(x) = \frac{2}{x}$. Note that $f(-1) = -2$ and $f(1) = 2$. Does $f(x)$ have a zero on $[-1, 1]$?