Linear Functions and Average Rate of Change

Section 2

Math 111, W22

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  - You are feeling up for a challenge and wish to do the steepest trail. Which trail should you hike?

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- So you should take the Mount Scott trail!

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  - Elevation 1.2 miles from the trailhead is $c(1.2)$, so $c(1.2) = 7900$ feet.
  - Our earlier computation to figure out "steepness" was exactly $\frac{c(1.2)-c(0)}{1.2-0} = 583.33$ feet per mile.

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  - $\frac{m(2.5)-m(0)}{2.5-0} = 600$
- This process seems to tell us how steep the trail is.

Formal Definition

- This motivates the following definition.
- Definition: The *Average Rate of Change* of a function $Q = f(t)$ on the interval $[a, b]$ is

$$ARC_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

This rate of change is frequently expressed as

$$\frac{\Delta Q}{\Delta t}$$

where ΔQ is the total change in the output Q and Δt is the total change in the input t over the interval $[a, b]$.

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Increasing, Decreasing, and Constant Functions

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  - A trail is strictly increasing in elevation if, at all points on the trail, you are going up. (i.e. there aren't any stretches of trail where you aren't going up)
- We can rephrase these conditions as:
  - A trail is strictly increasing in elevation if every stretch of trail is going up.

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- So we can turn our previous intuition into something a little more formal...
  - A trail is strictly increasing if for every interval $[a, b]$ in the practical domain, the elevation at b is greater than the elevation at a.
- But we have a way of phrasing "the elevation at b" a little bit more mathematically
  - A trail is strictly increasing if for every interval $[a, b]$ in the practical domain, $f(b) > f(a)$.

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- Definition: A function $f(t)$ is $\begin{cases} \text{strictly increasing} \\ \text{strictly decreasing} \\ \text{constant} \end{cases}$ on the interval (c, d) as long as $\begin{cases} f(b) > f(a) \\ f(b) < f(a) \\ f(b) = f(a) \end{cases}$ for every pair of values $a < b$ in the interval $[c, d]$.

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- Definition: A *linear function* is any function with the property that for any pair of points on the graph of the function, the average rate of change between those points has the same value, a. This value is called the *slope* of the function. Such a function can always be written

$$f(t) = at + b$$

for some numbers a and b. The number b is called the *intercept*.

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  - The average rate of change on that interval is

$$ARC_{[c,d]} = \frac{f(d) - f(c)}{d - c} = \frac{(ad + b) - (ac + b)}{d - c}$$
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  - Since this average rate of change does not depend on the choice of c, d it is always constant, agreeing with what we said earlier.

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  - Let's say you know that a linear function has intercept b. This tells you that when the input is 0, the output will be b.
    - Example: if I'm looking at a linear function $a(l)$ with intercept -3, I know that $a(0) = -3$

Linear Functions

- How do we actually solve problems about linear functions?
- Most problems that you can be asked will give you some kind of information, then ask you to find the equation of a linear function that satisfies that information.
- So how do you find the equation of a linear function?
- Note that finding a linear function requires two pieces of information
  1. Slope
  2. Intercept

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    - Answer: -9 foxes per year

Linear Functions

- (Recall: For a point on a graph (t_1, y_1), $y_1 = f(t_1)$.)
- If you are given two points on the graph of the function f, say (t_1, y_1) and (t_2, y_2), then the slope is just the average rate of change between those points,

$$\frac{y_2 - y_1}{t_2 - t_1}.$$

  - Example: Say we know that $(-1, 5)$ and $(2, 3)$ are points on the graph of a linear function. The slope of that function is

$$\frac{3 - 5}{2 - (-1)} = \frac{-2}{3}.$$

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    - Answer: 15 inches
  - If you know the slope and you know a single point on the graph, you can solve for the intercept.
    - Example: Suppose that weight is a linear function of age with slope 1.1 pounds per month. If a ten month old child weighs 20 pounds, what is the intercept of that function?

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    - Example: Suppose that weight is a linear function of age with slope 1.1 pounds per month. If a ten month old child weighs 20 pounds, what is the intercept of that function?
    - Let's do it together.

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- Again, perhaps unsurprisingly, these things correspond to:
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- So, the question is, what do these things mean on a graph?

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  - Graph $\ell(t) = 2t - 1$ by using this table of values and connecting the dots.

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0	-1
1	1

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- Slope is a little trickier, but it's the amount that you change vertically, if you change the input by 1.
- If you know anything at all about slope, it's that the more positive the slope gets, the more steeply upwards the graph points. The more negative the slope gets, the more steeply downward the graph points.

Collinearity

- Definition: We say that a set of points (t_1, y_1), (t_2, y_2), ..., (t_n, y_n) is *collinear* if there is a vertical line or single linear function whose graph contains all n points, i.e. if a single line passes through all of the points.
- How do we find out if a set of points is collinear? Find the line that goes through two of the points, then check to see if the other points lie on that line!
  - Finding all of the possible lines, then checking to see if they're the same is a lot of work.

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- Example: Are the points $(2, 4)$, $(-4, -5)$, $(0, 1)$ collinear?

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- So using average rate of change, we should be able to tie linear and nonlinear functions together, in some sense.

- If we start with a nonlinear function, $n(x)$, we can compute the average rate of change between a, b by $\frac{n(b)-n(a)}{b-a}$, which is the exact same formula as the formula for slope between the points $(a, n(a))$ and $(b, n(b))$.

Graph of n:

Synthesis (continued)

- So the average rate of change between two points on the graph of a function is exactly the slope of the line between the two points.
- This is really important in calculus, where the entire goal is to approximate functions by straight lines.
- Every "nice" curve has the property that, if you zoom in far enough, it looks like a straight line. (Example: the Earth is actually curved, even though it looks flat from our perspective.)