Ramsey Theory

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These notes are heavily based on my undergraduate intro combinatorics "textbook," which was a collection of notes from various BSM professors and students.

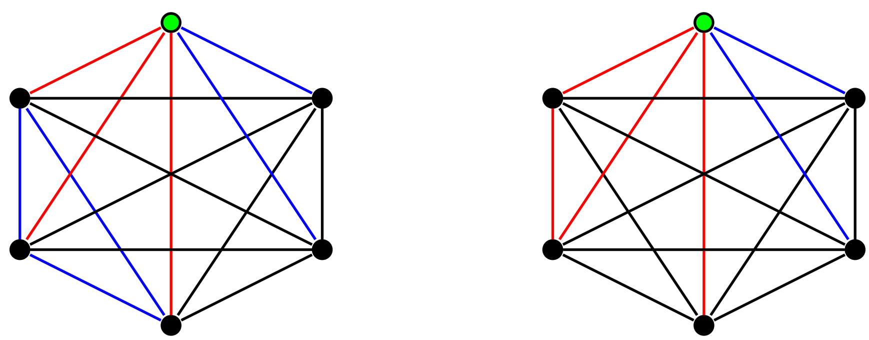
1 Party of Six

To break tradition, we start with a theorem.

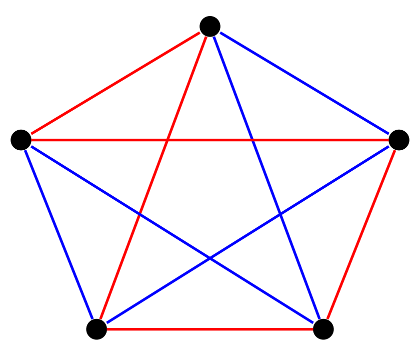
Theorem 1. Suppose 6 people are at a party. It is guaranteed that there are either

1. three people who all pairwise know each other, or
2. three people who are all pairwise strangers.

Proof. Pick one of the party-goers. They either know three other people or don't know three other people. WLOG, suppose they know three other people. Then those three people are either all strangers and we win, or some two of them know each other and then they form a group-of-knowing with the original party-goer, so we also win. □



Question: Can the same be said of parties of five?
Answer: No. Here's a counterexample!



More generally, our question of the day is "how big does a structure have to be to be sure that it contains a certain smaller structure?". Let's formalize this.

2 Ramsey numbers

Definition. The **graph Ramsey number** $R(G_1, G_2)$ is the smallest n such that for every graph G on n vertices, either G contains a subgraph isomorphic to G_1 or $\overline{G}$ contains a subgraph isomorphic to G_2. If $G_1 = K_k$ and $G_2 = K_\ell$ (the complete graphs on k and ℓ vertices respectively) and $k, \ell \geq 2$, then we denote the corresponding Ramsey number by $R(k, \ell)$.

Theorem 2 (Ramsey's Theorem)**.** *The Ramsey number $R(k, \ell)$ for $k, \ell \geq 2$ exists.*

Example. Our introductory problem about a party of six people shows that $R(3, 3) = 6$.

Example. Observe that $R(2, k) = R(k, 2) = k$.

Example. This table contains the value of some small Ramsey numbers.

k, ℓ	2	3	4	5	6
2	2	3	4	5	6
3	3	6	9	14	20*
4	4	9	18		
5	5	14			
6	6	20*			

The starred values are only upper bounds.

Example. We will determine the graph Ramsey number $R(P_3, K_4)$. There are two things to check when computing a Ramsey number.

- We need to find a number n of vertices so that we can be assured any graph of that cardinality contains our given subgraph(s).
- We need to find a graph of size $n - 1$ that does not contain our desired subgraphs.

First, notice that if G does not contain P_3 then G cannot contain a vertex of degree two or more. So if G is maximal with respect to this property, that means it has $\frac{1}{2}|V(G)|$ pairwise disjoint edges. (Maximal means that adding any edge results in a graph containing a P_3.) Taking these maximal graphs with $|V(G)| \leq 6$, we can see that $\overline{G}$ does not contain a K_4. However, with $|V(G)| = 7$, then the unique maximal graph without P_3 is three edges and a singleton, and $\overline{G}$ contains K_4 in this case. Hence $R(P_3, K_4) = 7$.

Theorem 3. For every $k, \ell \geq 3$, the following inequality holds:

$$R(k, \ell) \leq R(k-1, \ell) + R(k, \ell - 1).$$

Proof. Take a graph G on $n = R(k-1, \ell) + R(k, \ell-1)$ vertices. Fix a vertex x. There must be either $R(k-1, \ell)$ vertices adjacent to x or $R(k, \ell-1)$ vertices **not** adjacent to x. Suppose we're in the former case. Consider the subgraph H spanned by the vertices adjacent to x (of which there are at least $R(k-1, \ell)$). By the definition of $R(k-1, \ell)$, H contains either

- a K_{k-1} which together with x and all edges between vertices in H and x form a K_k subgraph of G, or
- a K_ℓ in the complement, which means there is a K_ℓ in the complement of G as well.

The second case (in which there are $R(k, \ell-1)$ vertices **not** adjacent to x) is similar. $\square$

Theorem 4. We have $R(k, \ell) \leq \binom{k+\ell-2}{k-1} = \binom{k+\ell-2}{\ell-1}$.

Proof. (Sketch.) According to my notes, this proof can be done by induction. The base case is the first example in this section and the inductive step is the theorem above. $\square$

This theorem (sort of) completes the proof of Ramsey's Theorem from above.

3 History

Origin

Frank Ramsey was a British mathematician, philosopher, and economist born in 1903. He was incredibly bright and was a friend of Wittgenstein and Keynes. The theorem named after him was only a stepping stone lemma in a paper in logic. Due to liver problems, he died at 26.

Development

Esther Klein proposed a conjecture to George Szekeres that resulted in the following theorem.

Theorem 5 (Happy Ending Problem)**.** *Any set of five points in the plane in general position has a subset of four points that form the vertices of a convex quadrilateral.*

It was named this way by Erdös because it resulted in the marriage of Klein and Szekeres. This lead to the development of Ramsey theory. (The main idea of Ramsey theory being that a large enough structure must contain a certain smaller structure.)

4 How hard is it to compute big Ramsey numbers?

Answer: Pretty damn hard. Erdös is quoted (perhaps paraphrased?) in a 1990 Scientific American article about this topic.

> "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number [$R(5,5)$]. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for [$R(6,6)$], however, we would have no choice but to launch a preemptive attack."

A 2016 Scientific American article considers the effect of Moore's law on this idea. (Moore's law is approximated as "computation speed doubled every 18 months since 1990.") The author concludes that purely based on the size of these numbers (and without using mathematical techniques associated to Ramsey theory) Erdös's prediction is still accurate. In other words, Moore's law does not keep up with the difference between $R(5,5)$ and $R(6,6)$ yet.

We know $R(5,5)$ is between 43 and 49. If we naively search with computers, there are about 10^{298} graphs to check of size 45.

We know $R(6,6)$ is between 102 and 165. If we naively search with computers, there are about 10^{1550} graphs to check of size 102.

For context, there are about 10^{82} atoms in the universe.

5 Generalizations

Definition. Let $R_t(G_1, G_2, \ldots, G_t) = R(G_1, G_2, \ldots, G_t)$ denote the smallest n such that for every coloring of the edges of the complete graph K_n with t colors there will be a color, say color i, such that the i-colored edges will contain a subgraph isomorphic to G_i. Similarly to the original definition, we may define $R(k_1, k_2, \ldots, k_t)$ to be $R(K_1, K_2, \ldots, K_t)$.

Similarly to the classic case, we can bound these generalized Ramsey numbers above by multinomial coefficients. That will be ommited here.

Theorem 6. We have $R(3,3,3) = 17$.

Proof. Take a complete graph on 17 vertices with edges colored red, blue, or green. Choose any vertex x of it and focus on the 16 edge leaving this vertex. By the PHP, there will be at least six of them that are the same color, say blue. Consider the subgraph G induced by exactly the neighbors to x along these blue edges. If any of the edges of G are blue, then that edge plus the two edges connecting the ends of that edge to x form a blue triangle. Otherwise the edges of this six-vertex graph are all green and red. We know from our opening example that this means that there either a red triangle or a green triangle contained in this subgraph. Hence $R(3,3,3) \leq 17$.

It remains only to show that $R(3,3,3) > 16$. For a proof-by-picture, check out page 443 in "Graphs and Hypergraphs" by Berge. $\square$