An Introduction to Jack Symmetric Functions

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November 2023

Abstract

Jack symmetric functions are a one-parameter generalization of Schur symmetric functions. In the twenty minutes provided, I'll define Jack symmetric functions and talk about a particular basis for them. Then I'll describe why hook lengths are important to evaluating the scalar product on these objects.

Defining Jack Symmetric Functions

- Recall:
  - m_λ are monomial symmetric functions
  - p_λ are the power sum symmetric functions.
- Jack functions are a one parameter generalization of the Schur functions.
- Notation: We write $J_\lambda(x_1, x_2, \ldots; \alpha)$ for the Jack function corresponding to the partition λ.
- They are characterized by satisfying the following three conditions.
  - Orthogonality: $\langle J_\lambda, J_\mu \rangle = 0$ when $\lambda \neq \mu$ where the scalar product is defined by $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \alpha^{\ell(\lambda)}$.
  - Triangularity: Select coefficients $v_{\lambda\mu}(\alpha)$ so that

    $$J_\lambda = \sum_\mu v_{\lambda\mu}(\alpha) m_\mu.$$

    Then $v_{\lambda\mu}(\alpha) = 0$ unless $\mu \leq \lambda$ (dominance order). In other words, the c.o.b. matrix from the m basis to the J basis is upper triangular. *Uh, from what to what?*
  - Normalization: If $|\lambda| = n$, then the coefficient $v_{\lambda 1^n}$ of $x_1 x_2 x_3 \cdots x_n$ of J_n is $n!$.
- I won't prove this. (Stanley doesn't either.)
- Here are a couple down-to-earth examples.

  $$J_1(x_1, x_2, \ldots; \alpha) = x_1 + x_2 + x_3 + \cdots$$
  $$J_{1^n} = n! m_{1^n} = n! e_n$$

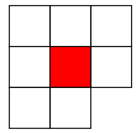
- Question: So how do we specialize to the Schur symmetric functions?
- Identify λ with its Young diagram.
- **Def'n:** The *hook length* of the box $\square \in \lambda$ is $h(\square) = a(\square) + \ell(\square) + 1$.

- **Def'n:** We write

$$H_\lambda = \prod_{\square \in \lambda} h(\square)$$

for the product of all the hook lengths of λ.

- **Example:** Let $\lambda = (3, 3, 2)$.



  - The hook length of the red box is 3.
  - Each box contains its hook length in the tableau below.

5	4	2
4	3	1
2	1	

  - So $H_\lambda = 5 \cdot 4 \cdot 2 \cdot 4 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 960$.

- Answer: $J_\lambda(x_1, x_2, x_3, \ldots; 1) = H_\lambda s_\lambda(x_1, x_2, x_3, \ldots)$.
- Proof (sketch): Stanley chapter 7 explains how s_λ satisfies the first two conditions. What is the coefficient of $x_1 x_2 \cdots x_n$ in s_λ? It's $f^\lambda = \frac{n!}{H_\lambda}$.
- Now let's talk about how to evaluate nontrivial scalar products. We'll need to define an alpha generalization of hook length.
- **Def'n:** We say $j_\lambda = \langle J_\lambda, J_\lambda \rangle$.
- **Def'n:** For a partition λ define

$$h^*_\lambda(\square) = \ell(\square) + \alpha(a(\square) + 1)$$
$$h^\lambda_*(\square) = 1 + \ell(\square) + \alpha \cdot a(\square).$$

In other words, arm boxes contribute an α to the hook length and leg boxes contribute a 1 to the hook length. The pivot box contribution is the difference between the upper and lower hook lengths. In the upper hook length the pivot gets an α and in the lower hook length it gets a 1.

- Note that this is the normal hook length when $\alpha = 1$.
- **Prop:** We have

$$j_\lambda = \prod_{\square \in \lambda} h^*_\lambda(\square) h^\lambda_*(\square).$$

- I won't prove this today.
- **Example:** Let $\mu = (3, 2)$. Then j_μ is the product of all of the expressions in the boxes below.

$3\alpha + 1$ $2\alpha + 2$	$2\alpha + 1$ $\alpha + 2$	α 1
2α $\alpha + 1$	α 1	